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Aspects of holographic entanglement entropy

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Abstract: This is an extended version of our short report [1], where a holographic interpretation of entanglement entropy in conformal field theories is proposed from AdS/CFT correspondence. In addition to a concise review of relevant recent progresses of entanglement entropy and details omitted in the earlier letter, this paper includes the following several new results: We give a more direct derivation of our claim which relates the entanglement entropy with the minimal area surfaces in the $\mathrm{AdS}_3/\mathrm{CFT}_2$ case as well as some further discussions on higher dimensional cases. Also the relation between the entanglement entropy and central charges in 4D conformal field theories is examined. We check that the logarithmic part of the 4D entanglement entropy computed in the CFT side agrees with the AdS_5 result at least under a specific condition. Finally we estimate the entanglement entropy of massive theories in generic dimensions by making use of our proposal.

Keywords: Black Holes in String Theory, Brane Dynamics in Gauge Theories, Gauge-gravity correspondence, AdS-CFT Correspondence.

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Contents

1. Introduction

When we study properties of a given quantum field theory (QFT), it is common to first investigate behaviors of correlation functions of local operators in the theory. However, properties of non-local quantities are equally important, especially for understanding of its quantum mechanical phase structure. One basic such example of non-local physical quantities is the Wilson loop operators in gauge theories, which is a very useful order parameter of confinement.

In a more generic class of QFTs, we can instead consider a quantity called entanglement entropy (or geometric entropy). This is defined as the von Neumann entropy S_A when we 'trace out' (or smear out) degrees of freedom inside a d-dimensional space-like submanifold B in a given $d+1$ dimensional QFT. Its complement is denoted by the submanifold A. It measures how a given quantum system is entangled or strongly correlated. Intuitively we can also say that this is the entropy for an observer in A who is not accessible to B as the information is lost by the smearing out in region B.

As its name suggests, we expect that the entanglement entropy is directly related to the degrees of freedom. Indeed, the entanglement entropy is proportional to the central charge in two dimensional conformal field theories (2D CFTs) as first pointed out in [2]. Recently, this property was also confirmed in [3] in which a general prescription of computing the entropy in 2D CFTs is given. Also in the mass perturbed CFTs (massive QFTs) the same conclusion holds [4, 5, 3]. Furthermore, as we will discuss later, the similar statement is also true in 4D CFTs. The entropy is related to the 4D central charges.

In higher dimensional (more than two dimensional) QFTs, it is not easy to compute the entanglement entropy for arbitrary submanifolds A even in free field theories. Motivated by this, we would like to consider a holographic estimation of the quantity by applying AdS/CFT correspondence (or duality) [6, 7]. We can find pioneering works [8, 9] that discuss related issues from slightly different viewpoints. Recently, the authors of the present paper proposed a holographic computation of entanglement entropy in CFTs from the

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AdS/CFT [1]. This reduces the complicated quantum calculation in QFTs to a classical differential geometrical computation.

The AdS/CFT correspondence relates a $d+2$ dimensional AdS space (AdS_{d+2}) to a $d+1$ dimensional CFT (CFT_{d+1}), which is sitting at the boundary of the AdS_{d+2}. The claim is that the entropy S_A in a $d+1$ dimensional CFT can be determined from the d dimensional minimal surface γ_A whose boundary is given by the $d-1$ dimensional manifold $\partial\gamma_A = \partial A$. The entropy is given by applying the Bekenstein-Hawking entropy formula to the area of the minimal surface γ_A as if γ_A were an event horizon. This is motivated by the idea of the entropy bound [10−12] and by the similarity between the black hole horizons and the minimal surface γ_A. They become equivalent in the special cases such as those in AdS black holes [1] and in black holes of brane-world [13], as the minimal surfaces wrap the horizons (see also [8, 9, 14] for earlier related discussions[1]). In [1] we have shown that our proposal, when applied to the lowest dimensional $\mathrm{AdS}_3/\mathrm{CFT}_2$ example, correctly reproduces the known results of entanglement entropy in 2D CFT. Also it is easy to see that the Bekenstein-Hawking like formula is consistent with the known 'area law' in entanglement entropy [16, 17] for the CFTs (also QFTs) in any dimensions.

In the present paper we would like to study the entanglement entropy in higher dimensional CFTs, especially CFT_4 from both the CFT and gravity sides. In particular, we find the computations of the logarithmic term from both sides agree at least when the second fundamental form of ∂A embedded in the d dimensional space vanishes. In addition, we present a review of the required knowledge of the entanglement entropy in conformal field theories and the details of our short report.

We would also like to mention recent interests in entanglement entropy in condensed matter physics. One of main foci in modern condensed matter physics is to understand quantum phases of matter which are beyond the Ginzburg-Landau paradigm. Many-body wavefunctions of quantum ground states in these phases look featureless when one looks at correlation functions of local operators; They cannot be characterized by classical order parameters of some kind. Indeed, they should be distinguished by their pattern of entanglement rather than their pattern of symmetry breaking [18]. Thus, the entanglement entropy is potentially useful to characterize these exotic phases.

Indeed, this idea has been pushed extensively in recent couple of years for several 1D quantum systems. It has been revealed that several quantum phases in 1D spin chains, including Haldane phases, can be distinguished by different scaling of the entanglement entropy. See, for example, [4, 5, 19−21] and references in [3].

For higher dimensional condensed matter systems, there has been not many works in this direction yet. Recently, the entanglement entropy was applied for so-called topological phases in 2+1 D [22, 23]. Typically, these phases have a finite gap and are accompanied by many exotic features such as fractionalization of quantum numbers, non-Abelian statistics of quasi-particles, topological degeneracy, etc. They can be also useful fault tolerant quantum computations.

[1] Also refer to [15] for recent arguments on the relation between the entanglement entropy in three qubit systems and the entropy of BPS black holes, based on similarities of their symmetries, though this does not seems to be related to our issues directly.

On the other hand, unconventional quantum liquid phases with gapless excitations, such as gapless spin liquid phases, seem to be, at least at present, more difficult to characterize in higher dimensions. Our results from AdS/CFT correspondence can be useful to study these gapless spin liquid states (some of these phases have been suspected to be described by a relativistic gauge field theory of some sort [18]) .

The organization of the present paper is as follows. In section 2 we present a review of definition and basic properties of entanglement entropy. Section 3 is devoted to computations of entanglement entropy in 2D CFTs. In section 4 we first summarize the known facts on entanglement entropy in higher dimensional CFTs and perform explicit computations especially for 4D CFTs. Next we relate the central charges in a given 4D CFT to its entanglement entropy. In section 5 we present our proposal of holographic computations of entanglement entropy from AdS/CFT. We also give an explicit proof of this claim in $\mathrm{AdS}_3/\mathrm{CFT}_2$ based on the well-known relation [24, 25] and discuss its extension to higher dimensional cases. Based on our proposal, in section 6, we compute the entanglement entropy in 2D CFTs from the AdS_3 side and find agreements. Higher dimensional cases are considered in section 7 where we compute the entropy from the analysis of AdS_{d+2} spaces. We compare it with the CFT results especially for $\mathrm{AdS}_5/\mathrm{CFT}_4$ case and find an agreement under a specific condition for simplification. We also estimate entanglement entropy in massive or non-conformal theories. In section 8 we summarize our results and discuss future problems.

2. Basics of entanglement entropy

We start with a review of basic ideas and properties of entanglement entropy.

2.1 Definition of entanglement entropy

Consider a quantum mechanical system with many degrees of freedom such as spin chains. More generally, we can consider arbitrary lattice models or QFTs including CFTs. We put the system at zero temperature and then the total quantum system is described by the pure ground state $|\Psi\rangle$. We assume no degeneracy of the ground state. Then, the density matrix is that of the pure state

$$\rho_{\mathrm{tot}} = |\Psi\rangle\langle\Psi|. \tag{2.1}$$

The von Neumann entropy of the total system is clearly zero $S_{\mathrm{tot}} = -\mathrm{tr}\,\rho_{\mathrm{tot}} \log \rho_{\mathrm{tot}} = 0$.

Next we divide the total system into two subsystems A and B. In the spin chain example, we just artificially cut off the chain at some point and divide the lattice points into two groups. Notice that physically we do not do anything to the system and the cutting procedure is an imaginary process. Accordingly the total Hilbert space can be written as a direct product of two spaces $\mathcal{H}_{\mathrm{tot}} = \mathcal{H}_A \otimes \mathcal{H}_B$ corresponding to those of subsystems A and B. The observer who is only accessible to the subsystem A will feel as if the total system is described by the reduced density matrix ρ_A

$$\rho_A = \mathrm{tr}_B\ \rho_{\mathrm{tot}}, \tag{2.2}$$

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where the trace is taken only over the Hilbert space $\mathcal{H}_B$.

Now we define the entanglement entropy of the subsystem A as the von Neumann entropy of the reduced density matrix ρ_A

$$S_A = -\mathrm{tr}_A\, \rho_A \log \rho_A. \tag{2.3}$$

This quantity provides us with a convenient way to measure how closely entangled (or how "quantum") a given wave function $|\Psi\rangle$ is. Notice also that in time-dependent backgrounds the density matrix ρ_{tot} and ρ_A are time dependent as dictated by the von Neumann equation. Thus we need to specify the time $t = t_0$ when we measure the entropy. In this paper, we always study static systems and we can neglect this issue.

It is also possible to define the entanglement entropy $S_A(\beta)$ at finite temperature $T = \beta^{-1}$. This can be done just by replacing (2.1) with the thermal one $\rho_{\mathrm{thermal}} = e^{-\beta H}$, where H is the total Hamiltonian. When A is the total system, $S_A(\beta)$ is clearly the same as the thermal entropy.

2.2 Properties

There are several useful properties which the entanglement entropy satisfies generally. We consider the zero temperature case. We summarize some of them as follows:

- (i) When B is the complement of A as before, we obtain

$$S_A = S_B. \tag{2.4}$$

  This manifestly shows that the entanglement entropy is not an extensive quantity. This equality (2.4) is violated at finite temperature.

- (ii) When A is divided into two subsystems A_1 and A_2, we find

$$S_{A_1} + S_{A_2} \geq S_A. \tag{2.5}$$

  This is called subadditivity.

- (iii) For any three subsystems A, B and C that do not intersect each other, the following strong subadditivity inequality holds:

$$S_{A+B+C} + S_B \leq S_{A+B} + S_{B+C}. \tag{2.6}$$

  Equivalently, we can have a more strong version of (2.5) as follows

$$S_A + S_B \geq S_{A\cup B} + S_{A\cap B}, \tag{2.7}$$

  for any subsystems A and B. When A and B do not intersect with each other, this relation is reduced to the subadditivity (2.5.)

More details of properties of the entanglement entropy can be found in e.g. [26].

2.3 Entanglement entropy in QFTs and area law

Consider a QFT on a $d+1$ dimensional manifold $\mathbb{R} \times N$, where $\mathbb{R}$ and N denote the time direction and the d dimensional space-like manifold, respectively. We define the subsystem by a d dimensional submanifold $A \subset N$ at fixed time $t = t_0$. We call its complement the submanifold B. The boundary of A, which is denoted by ∂A, divides the manifold N into two submanifolds A and B. Then we can define the entanglement entropy S_A by the previous formula (2.3). Sometimes this kind of entropy is called geometric entropy as it depends on the geometry of the submanifold A. Since the entanglement entropy is always divergent in a continuum theory we introduce an ultraviolet cut off a (or a lattice spacing). Then the coefficient in front of the divergence turns out to be proportional to the area of the boundary ∂A of the subsystem A as first pointed out in [16, 17],

$$S_A = \gamma \cdot \frac{\text{Area}(\partial A)}{a^{d-1}} + \text{subleading terms}, \tag{2.8}$$

where γ is a constant which depends on the system. This behavior can be intuitively understood since the entanglement between A and B occurs at the boundary ∂A most strongly. This result (2.8) was originally found from numerical computations [17, 16] and checked in many later arguments (see e.g. recent works [27 − 29]).

The simple area law (2.8), however, does not always describe the scaling of the entanglement entropy in generic situations. As we will discuss in detail in the later sections, the entanglement entropy of 1D quantum systems at criticality scales logarithmically with respect to the linear size l of A, $S_A \sim \frac{c}{3} \log l/a$ where c is the central charge of the CFT that describes the critical point. It has been also recently pointed out that the area law is corrected by a logarithmic factor as $S_A \propto (l/a)^{d-1} \log l/a+$ (subleading terms) for fermionic systems in the presence of a finite Fermi surface, where l is the characteristic length scale of the $d-1$ dimensional manifold ∂A [30 − 33]. Since we mainly consider relativistic QFTs (without a finite Fermi surface) in this paper, the area law (2.8) applies to our examples for $d \geq 2$ as we will see.

Before we proceed to further analysis of entanglement entropy, it might be interesting to notice that this area law (2.8) looks very similar to the Bekenstein-Hawking entropy (BH entropy) of black holes which is proportional to the area of the event horizon

$$S_{\text{BH}} = \frac{\text{Area of horizon}}{4G_N}, \tag{2.9}$$

where G_N is the Newton constant. Intuitively, we can regard S_A as the entropy for an observer who is only accessible to the subsystem A and cannot receive any signals from B. In this sense, the subsystem B is analogous to the inside of a black hole horizon for an observer sitting in A, i.e., outside of the horizon. Indeed, this similarity was an original motivation of the entanglement entropy [16, 17] (earlier related idea can also be found in [34]). Even though this analogy is not completely correct as it is, the one-loop quantum correction to the BH entropy in the presence of matter fields is known to be equal to the entanglement entropy [35]. This interesting relation is an important hint to find the holographic dual of the entanglement entropy discussed later. Indeed, the connection

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between this relation and our proposal has been found recently in [13] by employing the brane-world setup instead of AdS backgrounds.

3. Entanglement entropy in 2D CFT

Here we review and slightly extend existing computations of entanglement entropy in (1+1) D CFTs. The central charge of a given CFT is denoted by c. Such a computation was initiated in [2, 36] and a general prescription how to calculate the quantity was given in a recent work [3] (see also [37]), which we will explain in an orbifold theoretic manner. We separately discuss this lowest dimensional CFT since only in this case we can exactly compute the entropy for general systems at present.

3.1 How to compute entanglement entropy

In order to find the entanglement entropy, we first evaluate $\mathrm{tr}_A\,\rho_A^n$, differentiate it with respect to n and finally take the limit $n \to 1$ (remember that ρ_A is normalized such that $\mathrm{tr}_A\,\rho_A = 1$)

$$S_A = \lim_{n\to 1} \frac{\mathrm{tr}_A\,\rho_A^n - 1}{1-n} \tag{3.1}$$

$$= -\frac{\partial}{\partial n}\mathrm{tr}_A\,\rho_A^n|_{n=1} = -\frac{\partial}{\partial n}\log \mathrm{tr}_A\;\rho_A^n|_{n=1}. \tag{3.2}$$

This is called the replica trick. Therefore, what we have to do is to evaluate $\mathrm{tr}_A\;\rho_A^n$ in our 2D system. The first line of the above definition (3.1) without taking the $n \to 1$ limit defines the so-called Tsallis entropy, $S_{n,\mathrm{Tsallis}} = \frac{\mathrm{tr}_A\,\rho_A^n - 1}{1-n}$. [2]

This can be done in the path-integral formalism as follows. We first assume that A is the single interval $x \in [u, v]$ at $t_E = 0$ in the flat Euclidean coordinates $(t_E, x) \in \mathbb{R}^2$. The ground state wave function Ψ can be found by path-integrating from $t_E = -\infty$ to $t_E = 0$ in the Euclidean formalism

$$\Psi\left(\phi_0(x)\right) = \int_{t_E=-\infty}^{\phi(t_E=0,x)=\phi_0(x)} D\phi\; e^{-S(\phi)}, \tag{3.3}$$

where $\phi(t_E, x)$ denotes the field which defines the 2D CFT. The values of the field at the boundary ϕ_0 depends on the spacial coordinate x. The total density matrix ρ is given by two copies of the wave function $[\rho]_{\phi_0\phi_0'} = \Psi(\phi_0)\bar{\Psi}(\phi_0')$. The complex conjugate one $\bar{\Psi}$ can be obtained by path-integrating from $t_E = \infty$ to $t_E = 0$. To obtain the reduced density matrix ρ_A, we need to integrate ϕ_0 on B assuming $\phi_0(x) = \phi_0'(x)$ when $x \in B$.

$$[\rho_A]_{\phi_+\phi_-} = (Z_1)^{-1}\int_{t_E=-\infty}^{t_E=\infty} D\phi\; e^{-S(\phi)} \prod_{x\in A} \delta\left(\phi(+0,x) - \phi_+(x)\right)\cdot\delta\left(\phi(-0,x)-\phi_-(x)\right), \tag{3.4}$$

[2]The Tsallis entropy is related to the alpha entropy (Rényi entropy) $S_\alpha = \frac{\log \mathrm{tr}_A\,\rho_A^\alpha}{1-\alpha}$ through $S_{\alpha,\mathrm{Tsallis}} = \frac{1}{1-\alpha}[e^{(1-\alpha)S_\alpha} - 1]$ [20]. The $\alpha \to 1$ and $\alpha \to \infty$ limits of the alpha entropy give the von Neumann entropy and the single-copy entanglement entropy, respectively.

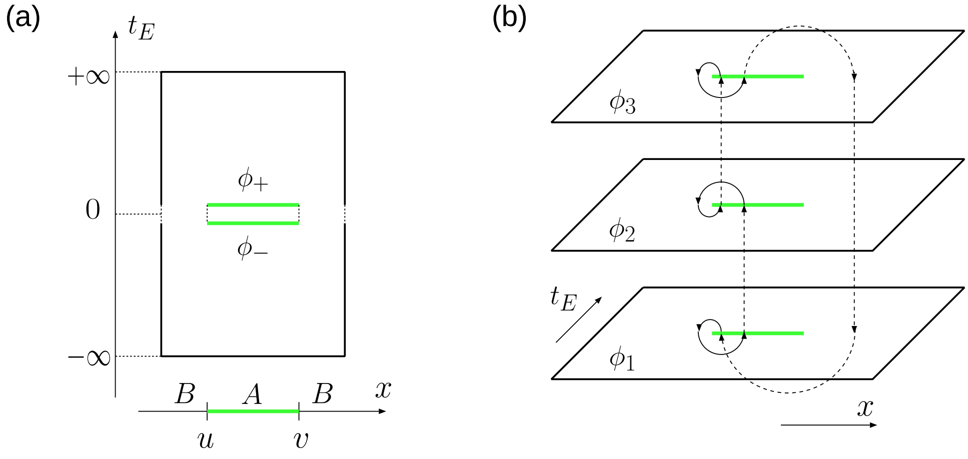

Figure 1: (a) The path integral representation of the reduced density matrix $[\rho_A]_{\phi_+\phi_-}$. (b) The n-sheeted Riemann surface $\mathcal{R}_n$. (Here we take $n = 3$ for simplicity.)

where Z_1 is the vacuum partition function on $\mathbb{R}^2$ and we multiply its inverse in order to normalize ρ_A such that $\mathrm{tr}_A\, \rho_A = 1$. This computation is sketched in figure 1 (a).

To find $\mathrm{tr}_A\, \rho_A^n$, we can prepare n copies of (3.4)

$$[\rho_A]_{\phi_{1+}\phi_{1-}}[\rho_A]_{\phi_{2+}\phi_{2-}} \cdots [\rho_A]_{\phi_{n+}\phi_{n-}}, \tag{3.5}$$

and take the trace successively. In the path-integral formalism this is realized by gluing $\{\phi_{i\pm}(x)\}$ as $\phi_{i-}(x) = \phi_{(i+1)+}(x)$ $(i = 1, 2, \cdots, n)$ and integrating $\phi_{i+}(x)$. In this way, $\mathrm{tr}_A\, \rho_A^n$ is given in terms of the path-integral on an n-sheeted Riemann surface $\mathcal{R}_n$ (see figure 1 (b))

$$\mathrm{tr}_A\, \rho_A^n = (Z_1)^{-n} \int_{(t_E,x)\in\mathcal{R}_n} D\phi\; e^{-S(\phi)} \equiv \frac{Z_n}{(Z_1)^n}. \tag{3.6}$$

To evaluate the path-integral on $\mathcal{R}_n$, it is useful to introduce replica fields. Let us first take n disconnected sheets. The field on each sheet is denoted by $\phi_k(t_E, x)$ $(k = 1, 2, \cdots, n)$. In order to obtain a CFT on the flat complex plane $\mathbb{C}$ which is equivalent to the present one on $\mathcal{R}_n$, we impose the twisted boundary conditions

$$\phi_k(e^{2\pi i}(w-u)) = \phi_{k+1}(w-u), \quad \phi_k(e^{2\pi i}(w-v)) = \phi_{k-1}(w-v), \tag{3.7}$$

where we employed the complex coordinate $w = x + it_E$. Equivalently we can regard the boundary condition (3.7) as the insertion of two twist operators $\Phi_n^{+(k)}$ and $\Phi_n^{-(k)}$ at $w = u$ and $w = v$ for each ($k-$th) sheet. Thus we find

$$\mathrm{tr}_A\, \rho_A^n = \prod_{k=0}^{n-1} \langle \Phi_n^{+(k)}(u) \Phi_n^{-(k)}(v) \rangle. \tag{3.8}$$

3.2 Derivation of entanglement entropy in an infinitely long system

When ϕ is a real scalar field, this is a non-abelian orbifold. To make the situation simple, assume that ϕ is a complex scalar field. Then we can diagonalize the boundary condition by defining n new fields $\tilde{\phi}_k = \frac{1}{n}\sum_{l=1}^{n} e^{2\pi i l k/n}\phi_l$. They obey the boundary condition

$$\tilde{\phi}_k(e^{2\pi i}(w-u)) = e^{2\pi i k/n}\tilde{\phi}_k(w-u), \quad \tilde{\phi}_k(e^{2\pi i}(w-v)) = e^{-2\pi i k/n}\tilde{\phi}_k(w-v). \tag{3.9}$$

Thus in this case we can conclude that the system is equivalent to $n-$disconnected sheets with two twist operators $\sigma_{k/n}$ and $\sigma_{-k/n}$ inserted in the $k-$th sheet for each values of k. In the end we find

$$\mathrm{tr}_A\, \rho_A^n = \prod_{k=0}^{n-1}\langle\sigma_{k/n}(u)\sigma_{-k/n}(v)\rangle \sim (u-v)^{-4\sum_{k=0}^{n-1}\Delta_{k/n}} = (u-v)^{-\frac{1}{3}(n-1/n)}, \tag{3.10}$$

where $\Delta_{k/n} = -\frac{1}{2}\left(\frac{k}{n}\right)^2 + \frac{1}{2}\frac{k}{n}$ is the (chiral) conformal dimension of $\sigma_{k/n}$. When we have m such complex scalar fields we simply obtain

$$\mathrm{tr}_A\, \rho_A^n = \prod_{k=0}^{n-1}\langle\sigma_{k/n}(u)\sigma_{-k/n}(v)\rangle \sim (u-v)^{-\frac{c}{6}(n-1/n)}, \tag{3.11}$$

setting the central charge $c = 2m$.

To deal with a general CFT with central charge c, we need to go back to the basis (3.7). The paper [3] showed that the result (3.11) is generally correct (see also [38]). The argument is roughly as follows. Define the coordinate z as follows

$$z = \left(\frac{w-u}{w-v}\right)^{\frac{1}{n}}. \tag{3.12}$$

This maps $\mathcal{R}_n$ to the z-plane $\mathbb{C}$. In this simple coordinate system we easily find $\langle T(z)\rangle_{\mathbb{C}} = 0$. Via Schwartz derivative term in the conformal map we obtain a non-vanishing value of $\langle T(w)\rangle_{\mathcal{R}_n}$. From that result, we can learn that twist operators $\Phi_n^{\pm(k)}$ in (3.8) have conformal dimension $\Delta_n = \frac{c}{24}(1-n^{-2})$. Thus we find the same result (3.11) for general CFTs as follows from (3.8).

Applying the formula (3.2) to (3.11), we find[3] the famous result [2]

$$S_A = \frac{c}{3}\,\log\frac{l}{a}, \tag{3.13}$$

where a is the UV cut off (or lattice spacing) and we set $l \equiv v-u$.

It is possible to extend the above result to the general case where A consists of multi intervals

$$A = \{w|\mathrm{Im}\, w = 0, \mathrm{Re}\, w \in [u_1,v_1]\cup[u_2,v_2]\cup\cdots\cup[u_N,v_N]\}. \tag{3.14}$$

We obtain the value of the trace [3]

$$\mathrm{tr}_A\, \rho_{A_w}^n \sim \left(\frac{\prod_{1\le j<k\le N}(u_k-u_j)(v_k-v_j)}{\prod_{j,k=1}^{N}(v_k-u_j)}\right)^{\frac{c}{6}(n-1/n)}. \tag{3.15}$$

[3] Here we neglect a constant term which does not depend on l, L and a.

Thus the entanglement entropy is given as follows [3]

$$S_A = \frac{c}{3} \sum_{1 \le i,j \le N} \log \frac{u_i - v_j}{a} - \frac{c}{3} \sum_{1 \le i<j \le N} \log \frac{u_j - u_i}{a} - \frac{c}{3} \sum_{1 \le i<j \le N} \log \frac{v_j - v_i}{a}. \tag{3.16}$$

3.3 Derivation of entanglement entropy on a circle

We assume the space direction x is compactified as a circle of circumference L. The system A is defined by the subsystem A by the union

$$A = \{x | x \in [r_1, s_1] \cup [r_2, s_2] \cup \cdots \cup [r_N, s_N]\}, \tag{3.17}$$

where we assume $0 \le r_1 < s_1 < r_2 < s_2 < \cdots < r_N < s_N \le L$. This subsystem A is related to the previous one (3.14) via the conformal map

$$w = \tan\left(\frac{\pi w'}{L}\right). \tag{3.18}$$

This maps the previous n-sheeted Riemann surface $w \in \mathcal{R}_n$ to the n-sheeted cylinder $w' \in \mathcal{C}yl_n$. We find $u_i = \tan\left(\frac{\pi r_i}{L}\right)$ and $v_i = \tan\left(\frac{\pi s_i}{L}\right)$.

To compute $\mathrm{tr}_A\, \rho^n_{A_{w'}}$ in this cylinder coordinates, we can apply the conformal transformations (3.18). This leads to the extra factor

$$\prod_{i=1}^{N} \left[\frac{L}{\pi} \cos\left(\frac{\pi r_i}{L}\right) \cos\left(\frac{\pi s_i}{L}\right)\right]^{-\frac{c}{6}(1-n^{-2})}, \tag{3.19}$$

which should be multiplied with (3.15). In this way, the entanglement entropy is given by

$$\begin{aligned} S_A =& \frac{c}{3} \sum_{1 \le i,j \le N} \log\left(\frac{L}{\pi a} \sin\left(\frac{\pi(r_i - s_j)}{L}\right)\right) \\ &- \frac{c}{3} \sum_{1 \le i<j \le N} \log\left(\frac{L}{\pi a} \sin\left(\frac{\pi(r_j - r_i)}{L}\right)\right) - \frac{c}{3} \sum_{1 \le i<j \le N} \log\left(\frac{L}{\pi a} \sin\left(\frac{\pi(s_j - s_i)}{L}\right)\right). \end{aligned} \tag{3.20}$$

When we only have one interval with the length l, (3.20) is reduced to the known result [2, 3]

$$S_A = \frac{c}{3} \cdot \log\left(\frac{L}{\pi a} \sin\left(\frac{\pi l}{L}\right)\right). \tag{3.21}$$

Notice that in the small l limit, (3.21) approaches to (3.13) as expected. Also the expression (3.21) is invariant under the exchange $l \to L - l$ and thus satisfies the property (2.4).

3.4 Derivation of entanglement entropy at finite temperature

It is also possible to calculate S_A at finite temperature $T = \beta^{-1}$ when its spacial length is infinite $L = \infty$. In this case we need to compactify the Euclidean time as $t_E \sim t_E + \beta$. We can map this system to the previous one (3.14) via the conformal map

$$w = e^{\frac{2\pi}{\beta} w'}. \tag{3.22}$$

We find $u_i = e^{\frac{2\pi r_i}{\beta}}$ and $v_i = e^{\frac{2\pi s_i}{\beta}}$. This conformal map leads to the extra factor

$$\prod_{i=1}^{N} \left[\frac{\beta}{2\pi} \, e^{-\frac{\pi}{\beta}(r_i+s_i)} \right]^{-\frac{c}{6}(1-n^{-2})}, \tag{3.23}$$

in addition to (3.15). Thus we obtain S_A as follows

$$\begin{aligned} S_A = &\frac{c}{3} \sum_{1\le i,j\le N} \log\left(\frac{\beta}{\pi a}\sinh\left(\frac{\pi(r_i-s_j)}{\beta}\right)\right) \\ &- \frac{c}{3} \sum_{1\le i<j\le N} \log\left(\frac{\beta}{\pi a}\sinh\left(\frac{\pi(r_j-r_i)}{\beta}\right)\right) - \frac{c}{3} \sum_{1\le i<j\le N} \log\left(\frac{\beta}{\pi a}\sinh\left(\frac{\pi(s_j-s_i)}{\beta}\right)\right). \end{aligned} \tag{3.24}$$

If the subsystem A is a single length l segment, it becomes the known result [3]

$$S_A = \frac{c}{3} \cdot \log\left(\frac{\beta}{\pi a}\sinh\left(\frac{\pi l}{\beta}\right)\right). \tag{3.25}$$

In the zero temperature limit $T \to 0$, this reduces to the previous result (3.13). On the other hand, in the high temperature limit $T \to \infty$, it approaches

$$S_A \simeq \frac{\pi c}{3} l T. \tag{3.26}$$

This is the same as the thermal entropy for the subsystem A as expected.

3.5 Massive theories

When we are away from a critical point, the logarithmic scaling law eq. (3.13) does not persist for $l > \xi$, where ξ is the correlation length (inverse of the mass gap). For large l ($\gg \xi$), the entanglement entropy saturates to a finite value [4, 3]

$$S_A = \mathcal{A} \cdot \frac{c}{6} \log \frac{\xi}{a}, \tag{3.27}$$

where $\mathcal{A}$ is the number of boundary points that separate A from its complement. Thus, unlike critical (1+1)D systems, the area law holds for the massive case. This behavior was studied in detail in several 1D quantum spin chains [4, 3, 21, 19], and QFTs [3, 37, 39]. In [3], the result (3.27) is derived from an argument similar to Zamolodchikov's c-theorem. We will mention this proof briefly in section 4.2.2.

4. Entanglement entropy in higher dimensional CFTs

Now we would like to move on to the computations of entanglement entropy in higher dimensional conformal field theories $\mathrm{CFT}_{d+1\ge 3}$. This was initiated in [16, 17] and a partial list of later results can be found in [40, 27, 3, 30, 31, 39, 28, 32, 41]. In spite of many progresses, the calculation of the entropy is too complicated to find exact results. This is one motivation to consider the holographic way of computing the quantity as we will discuss later.

As in the 2D CFT case explained in section 3, we assume the CFT_{d+1} is defined on the $d+1$ dimensional manifold $\mathbb{R}\times N$. We define the subsystem A as the submanifold of N at a fixed time $t=t_0\in\mathbb{R}$. The strategy of calculating the entanglement entropy S_A is the same as in the 2D case. First find the reduced trace $\mathrm{tr}_A\rho_A^n$ and then plug this in (3.2) to obtain S_A. We can compute $\mathrm{tr}_A\rho_A^n$ from the partition function Z_n on the n-sheeted $d+1$ dimensional manifold M_n as in the 2D case (3.6)

$$\mathrm{tr}_A\rho_A^n=\frac{Z_n}{(Z_1)^n}. \tag{4.1}$$

The n-sheeted manifold M_n can be constructed as follows. First we remove the infinitely thin d dimensional slice A from $M_1=\mathbb{R}\times N$. Then the boundary of such a space consists of two As, which we call A_{up} and A_{down}. Next we prepare n copies of such a manifold. Their boundaries are denoted by A^i_{up} and A^i_{down} $(i=1,2,\cdots,n)$. Now we glue A^i_{up} with A^{i+1}_{down} for every i. As we take the trace of ρ_A^n, $A^{i=n}_{\mathrm{up}}$ is glued with A^1_{down}. In the end this procedure leads to a manifold M_n with conical singularities where all n cuts meet.

It is not straightforward to calculate Z_n for an arbitrary choice of A even in free field theories. This is because the conformal structure is not as strong as in the 2D CFT case. Thus below we mainly restrict our arguments to specific forms of A given by the following two examples. We also simply assume $N=\mathbb{R}^d$.

The first one is the straight belt of width l

$$A_S=\{x_i|x_1\in[-l/2,l/2],x_{2,3,\cdots,d}\in[-\infty,\infty]\}, \tag{4.2}$$

as depicted in figure 2. Since the lengths in the directions of $x_2,x_3,\cdots,x_d$ are infinite, we often put the regularized length L. Taking the limit $l\to\infty$ and looking at the region near $x_1=-l/2$, we obtain the subsystem A_{SL} which covers a half infinite space of $\mathbb{R}^d$. The boundary in this case is given by the straight surface $\partial A_{\mathrm{SL}}=\mathbb{R}^{d-1}$.

The second example is the circular disk A_D of radius l defined by

$$A_D=\{x_i|r\le l\}, \tag{4.3}$$

where $r=\sqrt{\sum_{i=1}^d x_i^2}$ (see figure 2.).

4.1 Entanglement entropy of $d+1$ D massless free fields

As an explicit example, we consider the entanglement entropy of $d+1$ dimensional CFT on $\mathbb{R}^{1,d}$ defined by massless free fields such as a massless scalar field or Dirac (or Majorana) fermion. This can be regarded as the infinite volume limit $L\to\infty$ of the CFT on $M=\mathbb{R}^{1,1}\times T^{d-1}$, where the volume of torus is L^{d-1}.

Because this theory is free, we can perform the dimensional reduction on T^{d-1} and obtain infinitely many two dimensional free massive theories whose masses are given by

$$m^2=\sum_{i=2}^d k_i^2=\left(\frac{2\pi}{L}\right)^2\cdot\sum_{i=2}^d n_i^2, \tag{4.4}$$

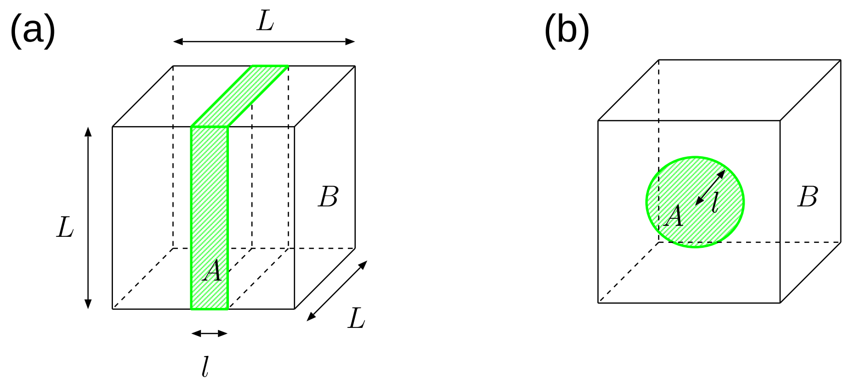

Figure 2: Two different shapes of the submanifold A considered in this paper. (a) The straight belt A_S and (b) the circular disk A_D. (Here, $d = 3$ for simplicity.)

where $k_i = \frac{2\pi n_i}{L}$ are the quantized momenta such that $n_i \in \mathbb{Z}$ in the torus directions.

To take this advantage we concentrate on the case where the subsystem A is defined by the straight belt A_S with radius l (4.2). The point is that the computation of the entanglement entropy S_A in this case is now reduced to the calculations of S_A in massive 2D QFTs.

4.1.1 Rough estimation

As we reviewed in section 3, we know the formulas of entanglement entropy both in the massless limit (i.e. $l \ll \xi$) (3.13) and the massive limit (3.27) (i.e. $l \gg \xi$). The correlation length is estimated as $\xi \sim m^{-1}$, where m is defined in (4.4). This leads to the following rough estimation of S_A by replacing the summations of infinitely many modes n_i with the integral of k_i in the $L \to \infty$ limit

$$\begin{aligned}
S_A^{\text{rough}} &= \sum_{k_2,\ldots,k_d}^{\xi \le l} \frac{c}{3} \log \frac{\xi}{a} + \sum_{k_2,\ldots,k_d}^{\xi \ge l} \frac{c}{3} \log \frac{l}{a} \\
&= \left(\frac{L}{2\pi}\right)^{d-1} \frac{c}{3} \left[\int_{l^{-1}}^{a^{-1}} d^{d-1}k \log \frac{\xi}{a} + \int_0^{l^{-1}} d^{d-1}k \log \frac{l}{a} \right] \\
&= \frac{c}{3(d-1) \cdot 2^{d-1} \pi^{\frac{d-1}{2}} \Gamma(\frac{d+1}{2})} \left[\frac{L^{d-1}}{a^{d-1}} - \frac{L^{d-1}}{l^{d-1}} \right].
\end{aligned} \tag{4.5}$$

If we set $d = 3$ (i.e. massless fields in 4 dimension), we obtain

$$S_A^{\text{rough}} = \frac{c}{24\pi} \left(\frac{L^2}{a^2} - \frac{L^2}{l^2} \right). \tag{4.6}$$

Notice that c is the two dimensional central charge and thus $c = 1$ for a 4D real scalar field and $c = 1$ (or $c = 2$) for a 4D Majorana (or Dirac) fermion. As can be seen from the exact computation discussed in the next subsection, this rough estimation already captures the correct functional form of the entanglement entropy.

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The first term in (4.5) represents the leading divergence which indeed obeys the area law (2.8). This part can be found by taking the limit $l \to \infty$ i.e. when A is the straight surface $A_{\rm SL}$. It is also possible to compute this term analytically as done in [40, 3]. On the other hand, the second term does not depend on the cutoff and thus is an interesting quantity to examine in more detail.

The violation of the area law for systems with a finite Fermi surface can be also understood from this rough estimation of the entanglement entropy. For simplicity, we assume a spherical Fermi surface with k_F being the Fermi momentum. For the momentum $\boldsymbol{k}$ outside and close to the Fermi surface, the gap is given by $m \sim \xi^{-1} \sim |\boldsymbol{k}| - k_F$. Thus, as before, the entanglement entropy is estimated as

$$S_A = \left(\frac{L}{2\pi}\right)^{d-1} \frac{c}{3} \left[\int_{|\boldsymbol{k}|=k_F+l^{-1}}^{a^{-1}} d^{d-1}k \log \frac{\xi}{a} + \int_0^{|\boldsymbol{k}|=k_F+l^{-1}} d^{d-1}k \log \frac{l}{a} \right]. \tag{4.7}$$

We thus find, for $l \to \infty$,

$$S_A \sim \frac{c}{3} \frac{2\pi^{\frac{d-1}{2}} k_F^{d-1}}{(d-1)\Gamma((d-1)/2)} \left(\frac{L}{2\pi}\right)^{d-1} \log \frac{l}{a} + \text{subleading terms}, \tag{4.8}$$

where note that $k_F \propto a^{-1}$. A more precise calculation based on the Widom conjecture in [31] gives the prefactor in front of $L^{d-1} \log l/a$ as the double integral over the fermi surface in the momentum space and the region ∂A in real space.

4.1.2 Exact estimation from entropic c-function

The previous approximation (4.5) uses the formulas which are exact only in the two opposite limits $\xi \to \infty$ and $\xi \to 0$. To perform an exact estimation, we need to be precise about the intermediate region $\xi \sim l$. In other words, we need to use a sort of c-function under the massive deformation corresponding to the interpolating region instead of the UV central charge in (4.5). To make this more explicit we can employ the entropic c-function C introduced in [42, 37, 39]. It is defined for 2D CFTs as follows

$$l \frac{dS_A(l)}{dl} = C(lm), \tag{4.9}$$

where l is the length of the subsystem A and m is the mass of the field. For massive free fermions and scalar fields, the function C is characterized as a solution to a differential equation of Painleve V type and its numerical form can be found in [37, 39]. Unfortunately, its analytical expression is not known.

This function $C(x)$ is positive and is also a monotonically decreasing function [42] with respect to x as in the Zamolodchikov's c-function [43]. These properties are indeed true in explicit examples [39], which we reproduced in figure 3 for a free massive real scalar boson and free Dirac fermion in 1+1 D. The function $C(x)$ is normalized such that in the UV limit $x = 0$ it is related to the ordinary central charge via $C(0) = c/3$. Note that if we set $C = C(0) = c/3$, we recover from this equation the well-known result (3.13). We will also show this later independently in (4.24). It was argued that the positivity of $C(x)$ is connected to a majorization relation for local density matrices [4, 5, 44–46]

In our example of the $d+1$ dimensional free field, we can reduce it to infinitely many massive fields in two dimensions. Thus in this case we again just have to sum over the discrete quantum numbers n_i. In the limit $L \to \infty$ we can replace the sum with an integral

$$\begin{aligned} l\frac{dS_A(l)}{dl} &= \left[\frac{L^{d-1}}{(2\pi)^{d-1}}\right]\int dk_2\cdots dk_d\ C(l|k|) \\ &= \left[\frac{L^{d-1}}{2^{d-2}\pi^{\frac{d-1}{2}}\Gamma\left(\frac{d-1}{2}\right)}\right]\int_0^\infty dk k^{d-2}C(lk). \end{aligned} \tag{4.10}$$

The merit of the quantity C instead of S_A itself is that it does not include UV divergences and thus we can set $a=0$ in C. After the integration of l we find

$$\begin{aligned} S_A(l) &= \left[\frac{L^{d-1}}{2^{d-2}\pi^{\frac{d-1}{2}}\Gamma\left(\frac{d-1}{2}\right)}\right]\int_0^\infty dk k^{d-2}\int_a^l \frac{d\tilde{l}}{\tilde{l}}C(\tilde{l}k), \\ &= \left(2^{d-1}\pi^{\frac{d-1}{2}}\ \Gamma\left((d+1)/2\right)\right)^{-1}\cdot\left[\int_0^\infty dx x^{d-2}C(x)\right]\cdot\left[\frac{L^{d-1}}{a^{d-1}}-\frac{L^{d-1}}{l^{d-1}}\right] \\ &\equiv K\left[\frac{L^{d-1}}{a^{d-1}}-\frac{L^{d-1}}{l^{d-1}}\right]. \end{aligned} \tag{4.11}$$

where we determine the integral constant by requiring that $S_A(l)$ should be vanishing[4] at $l=a$ since we are cutting off degrees of freedom below the energy scale a^{-1}. It is straightforward to find analogous formula for the free massive fields. This is given just by replacing k in (4.10) or (4.11) with $\sqrt{k^2+m^2}$.

The second term in (4.11) does not depend on the cutoff a. Thus we are interested in its coefficient K which is proportional to the integral of the function $x^{d-2}C(x)$. In principle, we can compute it numerically based on the numerical results of $C(x)$. Indeed by this method the coefficient K was computed for three dimensional free fields in [39]. We extend it to four dimensions which we are interested in later discussions and present the result as follows

$$K = \begin{cases} \dfrac{1}{\pi}\displaystyle\int_0^\infty \mathrm{d}t C(t) \simeq 0.039, & \text{for } d+1=3 \text{ dimensional real scalar boson} \\ \dfrac{1}{\pi}\displaystyle\int_0^\infty \mathrm{d}t C(t) \simeq 0.072, & \text{for } d+1=3 \text{ dimensional Dirac fermion} \\ \dfrac{1}{4\pi}\displaystyle\int_0^\infty \mathrm{d}t tC(t) \simeq 0.0049, & \text{for } d+1=4 \text{ dimensional real scalar boson} \\ \dfrac{1}{4\pi}\displaystyle\int_0^\infty \mathrm{d}t tC(t) \simeq 0.0097, & \text{for } d+1=4 \text{ dimensional Majorana fermion.} \end{cases} \tag{4.12}$$

To find the coefficient K in higher dimensions it is useful to notice that when x is large the entropic c-function $C(x)$ behaves as ($K_\nu(x)$ is the deformed Bessel function)

$$C_{\rm scalar}(x) \simeq \frac{1}{4}xK_1(2x), \quad \text{and} \quad C_{\rm Dirac}(x) \simeq \frac{1}{2}xK_1(2x), \tag{4.13}$$

[4] It is possible that this requirement is not absolute, i.e. this choice of the cutoff a may depend on the theory we consider. Thus only the constant K in front of the second term (i.e. finite term) in (4.11) has a qualitative meaning.

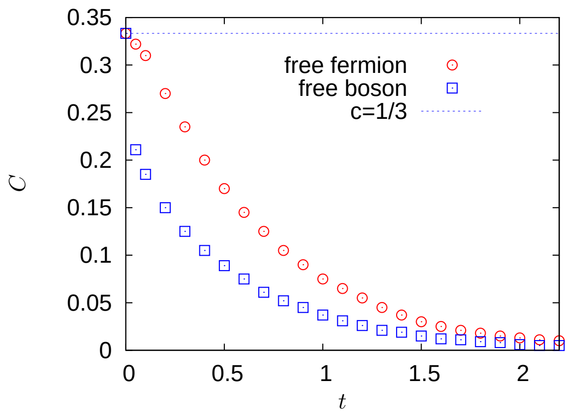

Figure 3: The entropic c-functions $C(x)$ for free massive real scalar boson and free Dirac fermion in 1+1 D reproduced from [39].

for a 2D free scalar field and a 2D Dirac fermion. When the dimension d is large, the contribution of the integral $\int dx x^{d-2} C(x)$ mainly comes from the large x region. Thus K can be well approximated by plugging (4.13) into (4.11). This leads to[5]

$$K_{\text{scalar}} \simeq 2^{-d-3} \pi^{(1-d)/2} \Gamma\left(\frac{d-1}{2}\right), \quad K_{\text{fermion}} \simeq 2K_{\text{scalar}}, \tag{4.14}$$

where K_{scalar} corresponds to a $d+1$ dimensional real scalar field while K_{fermion} to the $d+1$ dimensional fermions which is reduced to a 2D Dirac fermion[6].

Finally we would like to stress again that our result (4.11) was obtained by assuming a free field theory. In the presence of interactions we no longer have the simple sum over infinitely many massive fields in two dimensions (4.10) due to interactions between two different massive fields.

4.1.3 Entanglement entropy of 4D gauge field

As we will consider 4D gauge theories later, we would also like to examine the entanglement entropy of 4D gauge field. We neglect the interactions as before and thus we can concentrate on the abelian gauge theory. Its gauge fixed action with the ghost c is given by

$$S = \int d^4x \left[\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2\alpha} (\partial_\mu A^\mu)^2 + \bar{c}\,(-\Box)\, c\right]. \tag{4.15}$$

In order to compute the entanglement entropy, we consider the gauge theory on an n-sheeted manifold M_n as before. We can rewrite the gauge field action as follows (we fix

[5]For example, from this approximation we find $K_{\text{boson}} = 0.0497$ and $K_{\text{fermion}} = 0.00995$ for $d = 3$ and these are rather close to the previous results in (4.12). It may also be interesting to compare this with the our rough estimation done in the previous subsection. There we found $K^{\text{rough}}_{\text{scalar}} = K^{\text{rough}}_{\text{fermion}} = \frac{1}{24\pi} = 0.0133$ when $d = 3$.

[6]In higher dimension we need to multiply an appropriate degeneracy with K_{fermion} to obtain the result for a ordinary fermion such as Dirac fermion.

the gauge by setting $\alpha = 1$)

$$
\begin{aligned}
S_{A_\mu} &= \int d^4x \left[\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}\left(\partial_\mu A^\mu\right)^2\right] \\
&= \int d^4x \frac{1}{2}\partial_\mu A^\nu \partial^\mu A_\nu + \left[\int d^3x[-(\partial_\nu A^\nu)A^0 + (\partial_\mu A^0)A^\mu]\right]_{\mathrm{surf}},
\end{aligned} \tag{4.16}
$$

where $[\cdots]_{\mathrm{surf}}$ denotes the surface term when we performed an partial integration. If we neglect the surface term, the theory is equivalent to four real scalar fields and a ghost field which is a complex scalar. Since the complex ghost scalar field cancels two real scalars, the theory is equivalent to two real scalar fields. However, there is a subtle issue on the surface term that appears when we do the partial integration. Since fields are discontinuous along the time direction in some region of the spacetime, we got surface contributions. In this paper we assume such a term is not relevant for the computation of the entanglement entropy as[7] in [40].

4.2 Entanglement entropy and central charges in 4D CFT

As we have seen, the entanglement entropy in 2D CFTs is proportional to the central charge c. Since the central charge roughly measures the number of degrees of freedom N_{dof}, we find the entanglement entropy is also proportional to N_{dof}. This fact is very natural as its name of 'entropy' shows. Therefore we may expect that a similar story is true also in the higher dimensional theories. As such an example, below we consider 4D CFTs. Indeed we will find that an important part of the entanglement entropy is proportional to the central charges. See also [47, 41] for an earlier discussion.

In principle, it is possible to extend the relation between central charges and entanglement entropy to higher dimensions as far as the spacetime dimension is even. When we consider odd dimensional spacetime, we do not have any clear definition of central charges due to the absence of the Weyl anomaly. Under this situation, the entanglement entropy may play an important alternative role [8].

4.2.1 Entanglement entropy from Weyl anomaly

Central charges in CFTs can be defined from the Weyl anomaly (or conformal anomaly) $\langle T^\mu_\mu\rangle$. Define the energy-momentum tensor $T^{\mu\nu}$ in terms of the functional derivative of the (quantum corrected) action S with respect to the metric $g_{\mu\nu}$

$$
T^{\mu\nu} = \frac{4\pi}{\sqrt{g}}\frac{\delta S}{\delta g_{\mu\nu}}. \tag{4.17}
$$

In 2D CFTs, the Weyl anomaly is given by the well-known formula

$$
\langle T^\mu_\mu\rangle = -\frac{c}{12}R, \tag{4.18}
$$

[7] Indeed if we include such a contribution we find the total entropy of the gauge field becomes negative in the particular case discussed in [40], which looks strange if we remember the original definition (2.3).

[8] We are grateful to Anton Kapustin for pointing out this possibility to us.

where R is the scalar curvature. We can regard this as a definition of the central charge c in 2D CFTs.

Now we move on to 4D CFTs. In our normalization of (4.17), the Weyl anomaly can be written as

$$\langle T^\alpha_\alpha\rangle = -\frac{c}{8\pi}W_{\mu\nu\rho\sigma}W^{\mu\nu\rho\sigma} + \frac{a}{8\pi}\tilde{R}_{\mu\nu\rho\sigma}\tilde{R}^{\mu\nu\rho\sigma}. \tag{4.19}$$

in a curved metric background $g_{\mu\nu}$, where W and $\tilde{R}$ are the Weyl tensor and the dual of the curvature tensor. Notice that the second term is the Euler density. In terms of the ordinary curvature tensor, we can express the curvature square terms in (4.19) as follows

$$\begin{aligned} W_{\mu\nu\rho\sigma}W^{\mu\nu\rho\sigma} &= R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 2R_{\mu\nu}R^{\mu\nu} + \frac{1}{3}R^2, \\ \tilde{R}_{\mu\nu\rho\sigma}\tilde{R}^{\mu\nu\rho\sigma} &= R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 4R_{\mu\nu}R^{\mu\nu} + R^2. \end{aligned} \tag{4.20}$$

The coefficients c and a in (4.19) are called[9] the central charges of 4D CFTs [48–50]. This is the original definition of the central charges in 4D CFTs. The central charge a is believed to decrease monotonically under the renormalization group (RG) flow, while for c this is not true and indeed counter examples are known; these properties of the central charges a and c are confirmed in many supersymmetric examples e.g. [50].

To compute the entanglement entropy, we first consider the partition function Z_n on the $d+1$ dimensional n-sheeted manifold M_n. Then we find the trace of ρ^n reduced to the subsystem A is given by the formula (4.1). The entanglement entropy can be found by taking the derivative of n with the $n \to 1$ limit. If we define the length scale of the manifold A by l, then the scaling of l is related to the Weyl scaling. They should be the same[10] at least in the $n \to 1$ limit. In this way we find

$$\begin{aligned} l\frac{d}{dl}\log[\mathrm{tr}_A\,\rho_A^n] &= 2\int d^{d+1}x\; g_{\mu\nu}(x)\frac{\delta}{\delta g_{\mu\nu}(x)}\left[\log Z_n - n\log Z_1\right] \\ &= -\frac{1}{2\pi}\left\langle\int d^{d+1}x\;\sqrt{g}T^\mu_\mu(x)\right\rangle_{M_n} + \frac{n}{2\pi}\left\langle\int d^{d+1}x\;\sqrt{g}T^\mu_\mu(x)\right\rangle_{M_1}. \end{aligned} \tag{4.21}$$

When we consider a CFT on $M = \mathbb{R}^{d+1}$, the second term (i.e. integral on $M_1 = M = \mathbb{R}^{d+1}$) become obviously vanishing. Below we omit writing the second term explicitly just to make the appearance of expressions simple even if M is a curved manifold. Then the entanglement entropy satisfies

$$\begin{aligned} l\frac{d}{dl}S_A &= -\lim_{n\to 1} l\frac{d}{dl}\left(\frac{\partial}{\partial n}\log[\mathrm{tr}_A\,\rho_A^n]\right) \\ &= \frac{1}{2\pi}\lim_{n\to 1}\frac{\partial}{\partial n}\left\langle\int d^{d+1}x\;\sqrt{g}T^\mu_\mu(x)\right\rangle_{M_n}. \end{aligned} \tag{4.22}$$

[9]The central charge a should not be confused with a UV cutoff. To avoid confusion, a_{cutoff} is used to denote the UV cutoff in this subsection.

[10]Equally we can say that the scaling of l is oppositely related to the scaling of the cutoff i.e. $l\frac{d}{dl} = -a_{\mathrm{cutoff}}\cdot\frac{d}{da_{\mathrm{cutoff}}}$

4.2.2 Entanglement entropy and central charges

eq. (4.22) can be used to relate the entanglement entropy and central charge in a direct fashion. Let us apply (4.22) to 2D CFTs first. We assume the submanifold A is a segment of the length l in the total system. Then, the n-sheeted manifold M_n has two conical singularities at u and v that separate A and B. If one goes around these singularities, one picks up $2\pi n$ phase, i.e., $2\pi(n-1)$ extra phase compared with 2π. (See figure 1.) These singularities are reflected in the Euler number

$$\chi[M_n] = \frac{1}{4\pi}\int_{M_n} d^2x\sqrt{g}R = 2(1-n), \tag{4.23}$$

where we noted the scalar curvature is given by $R = 4\pi(1-n)[\delta^{(2)}(u) + \delta^{(2)}(v)]$ in the presence of a deficit angle $2\pi(1-n)$ at the conical singularities. Plugging (4.18) into (4.22), we obtain

$$l\frac{d}{dl}S_A = -\frac{\partial}{\partial n}\left(\frac{c}{24\pi}\int d^2x\sqrt{g}R\right) = \frac{c}{3}. \tag{4.24}$$

We thus reproduce the known result (3.13) (see also (4.9)).

It is also possible to derive (3.27) from (4.22) by noting that

$$m\frac{\partial S_A}{\partial m} = l\frac{\partial S_A}{\partial l} = \frac{1}{2\pi}\lim_{n\to 1}\frac{\partial}{\partial n}\left\langle\int d^2x\ \sqrt{g}T^\mu_\mu\right\rangle. \tag{4.25}$$

When $A = A_{\rm SL}$ (i.e., $\mathcal{A} = 1$ in (3.27)), the integral on the right hand side is evaluated as

$$\int d^2x\ \sqrt{g}\,\langle T^\mu_\mu\rangle = -\pi\frac{c}{6}\left(n - \frac{1}{n}\right), \tag{4.26}$$

by an argument similar to Zamolodchikov's c-theorem [3]. We thus recover (3.27).

If we repeat the same analysis in 4D CFTs, we find

$$l\frac{d}{dl}S_A = \lim_{n\to 1}\frac{\partial}{\partial n}\left[-\frac{c}{16\pi^2}\int_{M_n} d^4x\sqrt{g}W_{\mu\nu\rho\sigma}W^{\mu\nu\rho\sigma} + \frac{a}{16\pi^2}\int_{M_n} d^4x\sqrt{g}\tilde{R}_{\mu\nu\rho\sigma}\tilde{R}^{\mu\nu\rho\sigma}\right] \tag{4.27}$$

$$= \gamma_1\cdot\frac{\text{Area}(\partial A)}{a^2_{\text{cutoff}}} + \gamma_2, \tag{4.28}$$

where γ_1 and γ_2 are numerical constants. The first term in (4.28) comes from the integral of the W^2 term in (4.27) and represents the leading divergence $\sim a^{-2}_{cutoff}$. This is because the curvature tensor is divergent as $R \sim a^{-2}_{cutoff}$ at the surface ∂A, where the deficit angle presents and behaves like a delta function supported on ∂A. The Euler density term does not have such a divergence since it is a topological invariant. Thus the constant γ_1 is proportional to c. Another constant γ_2 comes from both terms in (4.27) and it is proportional to the linear combination of a and c. By integrating (4.28), we can express the entanglement entropy as follows

$$S_A = \frac{\gamma_1}{2}\cdot\frac{\text{Area}(\partial A)}{a^2_{\text{cutoff}}} + \gamma_2\log\frac{l}{a_{\text{cutoff}}} + S^{\text{others}}_A, \tag{4.29}$$

where the final term S_A^{others} expresses terms which are independent of the total scaling $l \to e^{\alpha} l$. In other words, S_A^{others} depends on the detailed shape of the surface ∂A. In this way, the central charges determine the entanglement entropy up to these contributions S_A^{others}. Notice that the leading divergence (4.29) agrees with the area law (2.8). In our later arguments using AdS/CFT duality, the gravity computations in section 7 reproduce the same behavior as (4.29). When we assume $a = c$, both γ_1 and γ_2 are proportional to a. This also agrees with our later gravity computations in section 7.3. For example, in the $\mathcal{N} = 4$ SU(N) super Yang-Mills the central charges are given by $a = c = (N^2 - 1)/4$ [51] and thus they satisfy the condition.

In particular, when A is the circular disk A_D with radius l, the system only depends on l and a_{cutoff}. Thus the trace anomaly completely determines the entanglement entropy S_A. On the other hand, in the case of the straight belt A_S there are two length scales l and L and the result (4.29) becomes less predictive. Indeed the finite term which we discussed before takes the form $\propto \frac{L^2}{l^2}$ and thus it is included in S_A^{others} in (4.29). Since this term is not directly related to the central charges, we expect that its value may be shifted when we change the t' Hooft coupling as is so in the thermal entropy. Indeed, the comparison of the numerical results from the AdS$_5\times$ S^5 and the free $\mathcal{N} = 4$ super Yang-Mills supports this speculation as we will see in section 7.3.

Even though the constant γ_1 is not universal in the sense that it depends on the choice of the UV cut off, the other one γ_2 is universal and an interesting quantity to evaluate. In principle, this is reduced to a differential geometric computation. Since the evaluation of total expression turns out to be rather complicated, below we would like to compute some particular important terms.

It is straightforward to evaluate the contribution from the second term (Euler density) in (4.27) because this is a topological term. As shown in [52], in a 4D manifold M_n with a codimension two surface Σ around which conical singularities develop (with a deficit angle $2\pi(1-n)$) we obtain

$$\chi[M_n] = \frac{1}{32\pi^2}\int_{M_n} d^4x\sqrt{g}\tilde{R}\tilde{R} = (1-n)\chi[\Sigma] + \frac{1}{32\pi^2}\int_{M_n-\Sigma} d^4x\sqrt{g}\tilde{R}\tilde{R}, \tag{4.30}$$

where $M_n - \Sigma$ denotes the smooth manifold defined by subtracting the singular part Σ from M_n. Therefore the contribution of the $\tilde{R}^2$ term in (4.27) to the constant γ_2 is given by

$$\gamma_2^{\text{top}} = -2a\cdot\chi[\partial A], \qquad (\text{especially,}\ \ \gamma_2^{\text{top}} = -4a \ \text{when}\ \ \partial A = \mathrm{S}^2). \tag{4.31}$$

To make the analysis of the W^2 term in (4.27) simple, below we only consider the case where the second fundamental form (or the extrinstic curvature) of ∂A, when embedded in the 4D manifold M_n, can be neglected. This is true when we consider the straight belt A_S. Another typical example is when M_n is an Euclidean black hole and ∂A is its horizon. We also concentrate on the case where ∂A is a connected manifold. Under these assumptions we can employ the differential geometric results in [52]

$$\begin{aligned}
&\int_{M_n} d^4x\sqrt{g}R^2 - \int_{M_n-\Sigma} d^4x\sqrt{g}R^2 = 8\pi(1-n)\int (R_\Sigma + 2R_{ii} - R_{ijij}) + \mathcal{O}((1-n)^2), \\
&\int_{M_n} d^4x\sqrt{g}R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma} - \int_{M_n-\Sigma} d^4x\sqrt{g}R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma} = 8\pi(1-n)\int R_{ijij} + \mathcal{O}((1-n)^2), \\
&\int_{M_n} d^4x\sqrt{g}R^{\mu\nu}R_{\mu\nu} - \int_{M_n-\Sigma} d^4x\sqrt{g}R^{\mu\nu}R_{\mu\nu} = 4\pi(1-n)\int R_{ii} + \mathcal{O}((1-n)^2),
\end{aligned} \tag{4.32}$$

where R_Σ is the intrinsic curvature of the 2D submanifold Σ; R_{ij} and R_{ijkl} denote the curvature tensors projected onto the direction normal to Σ (e.g. $R_{ij} = R_{\mu\nu}n_i^\mu n_j^\nu$ using the two orthonormal vectors $n_\mu^i \quad (i = 1, 2)$ orthogonal to Σ). In the end, we obtain[11] (this includes both contributions from W^2 and $\tilde{R}^2$)

$$\gamma_2 = \frac{c}{6\pi}\int_{\Sigma=\partial A} d^2x\sqrt{g}\,(R_{\Sigma=\partial A} + 2R_{ijij} - R_{ii}) - \frac{a}{2\pi}\int_{\Sigma=\partial A} d^2x\sqrt{g}R_{\Sigma=\partial A}. \tag{4.33}$$

Especially when $a = c$,

$$\gamma_2 = -\frac{a}{6\pi}\int_{\Sigma=\partial A} d^2x\sqrt{g}\,(2R_{\Sigma=\partial A} - 2R_{ijij} + R_{ii})\,. \tag{4.34}$$

under the previous assumption that the second fundamental form is zero. We will later compare this result with the one from gravity side in section 7.3.

5. Holographic interpretation

The main purpose of this paper is to compute the entanglement entropy in $d+1$ dimensional conformal field theories CFT_{d+1} via the AdS/CFT correspondence. This duality relates the CFT_{d+1} to the $d+2$ dimensional AdS space AdS_{d+2}. Then we expect that the entanglement entropy can be computed as a geometrical quantity in the AdS_{d+2} space just as the thermal entropy of CFTs is found from the area formula of AdS black hole entropy [54].

As in section 3 the CFT_{d+1} is defined on $M = \mathbb{R}\times N$ and we divide N into two regions A and B. We assume the space-like d dimensional manifold N is now given by $\mathbb{R}^d$ or S^d such that M is the boundary of AdS_{d+1} in the Poincare coordinates

$$ds^2 = R^2\,\frac{dz^2 - dx_0^2 + \sum_{i=1}^{d-1} dx_i^2}{z^2}, \tag{5.1}$$

or the global coordinates

$$ds^2 = R^2\left(-\cosh^2\rho dt^2 + d\rho^2 + \sinh\rho^2 d\Omega_d^2\right), \tag{5.2}$$

respectively.

[11] Refer also to [53] for an earlier computation of a similar expression of the logarithmic term from a different approach.

5.1 General proposal

In this setup we propose that the entanglement entropy S_A in CFT_{d+1} can be computed from the following area law relation

$$S_A = \frac{\mathrm{Area}(\gamma_A)}{4G_N^{(d+2)}}. \tag{5.3}$$

The manifold γ_A is the d-dimensional static minimal surface in AdS_{d+2} whose boundary is given by ∂A. Its area is denoted by $\mathrm{Area}(\gamma_A)$. Also $G_N^{(d+2)}$ is the $d+2$ dimensional Newton constant. It is obvious that the leading divergence $\sim a^{-(d-1)}$ in (5.3) is proportional to the area of the boundary ∂A and this agrees with the known property (2.8).

This proposal is motivated by the following physical interpretation. Since the entanglement entropy S_A is defined by smearing out the region B, the entropy is considered to be the one for an observer in A who is not accessible to B. The smearing process produces the fuzziness for the observer and that should be measured[12] by S_A. In the higher dimensional perspective of the AdS space, such an fussiness appears by hiding a part of the bulk space AdS_{d+2} inside an imaginary horizon, which we call γ. It is clear that γ covers the smeared region B from the inside of the AdS space and thus we find $\partial\gamma = \partial B (= \partial A)$. We expect that it is the holographic screen for the hidden part in the bulk. To choose the minimal surface as in (5.3) means that we are seeking the severest entropy bound [10−12] for the lost information. In the examples of $\mathrm{AdS}_3/\mathrm{CFT}_2$, we will show below that the bound is actually saturated. Therefore it is natural to expect that the bound is always saturated even in the higher dimensional ($d \geq 2$) cases. These considerations lead to our proposal (5.3). Notice also that the properties (2.4) and (2.5) are obviously satisfied for (5.3).

It is also straightforward to extend this formula (5.3) to any asymptotically AdS spaces and we argue that the claim remains the same in these generalized cases. For example, if we consider a AdS Schwarzschild black hole, then the minimal surface γ_A wraps the part of its real horizon as we will see later in section 7.5. This consideration fixes the normalization of (5.3).

5.2 How to understand the proposal from AdS/CFT

Let us try to understand how the area law (5.3) can be derived from known facts on AdS/CFT correspondence. As we have seen in section 3, it is essential to compute $\mathrm{tr}_A\, \rho_A^n$ in order to obtain the entanglement entropy. It is equivalent to the partition function of the CFT on the multiple (i.e. n times) covered space. Then S_A can be found from the formula (3.2).

Let us start with the $\mathrm{AdS}_3/\mathrm{CFT}_2$ example with a single interval. In this case as we have seen, $\mathrm{tr}_A\, \rho_A^n$ is equivalent to the n products of the two point functions $\langle \Phi_n^{+(k)} \Phi_n^{-(k)} \rangle$ as in (3.8). The conformal dimension of $\Phi_n^{(k)\pm}$ is given by $\Delta_n = \frac{c}{24}(1-n^{-2})$. The CFTs on disconnected n sheets (remember the description explained in section 3) is equivalent to a

[12] It may be interesting to note that this origin of entropy is somewhat analogous to the recently proposed 'fuzzball' picture (for a review see [55]).

CFT on a single sheet $\mathbb{R}^2$ whose central charge is nc with two twisted vertex operators Φ_n^+ and Φ_n^- (distinguish them from $\Phi_n^{(k)\pm}$) inserted.

In AdS/CFT[13], such a two point function $\langle \Phi_n^+(P)\Phi_n^-(Q)\rangle$ in the CFT can be computed as[14]

$$\langle \Phi_n^+(P)\Phi_n^-(Q)\rangle \sim e^{-\frac{2n\Delta_n \cdot L_{PQ}}{R}}, \tag{5.4}$$

where L_{PQ} is the geodesic distance between P and Q. Therefore we can derive explicitly the area law (5.3) as follows

$$S_A = 2\left(\left.\frac{\partial(n\Delta_n)}{\partial n}\right|_{n=1}\right) \cdot \frac{L_{\gamma_A}}{R} = \frac{L_{\gamma_A}}{4G_N^{(3)}}, \tag{5.5}$$

from AdS/CFT correspondence.

In higher dimensions, we can again compute $\mathrm{tr}_A\, \rho_A^n$ as the path-integral over the multi covered space with n sheets. We expect that this system is equivalent to a CFT on $\mathbb{R}^{1,d}$ which has the n replica fields $\phi_i(x^\mu)$ with a twist-like operator Φ_n inserted (assuming the simplest case that ∂A is a connected manifold). Notice that this operator is localized in codimension two subspace of $\mathbb{R}^{1,d}$. Then $\mathrm{tr}_A\, \rho_A^n$ is equal to the one point function $\langle \Phi_n \rangle$. As in the Wilson loop operator case [66], we naturally expect that it can be computed as

$$\langle \Phi_n \rangle \sim e^{-\alpha_n \mathrm{Area}_A}, \tag{5.6}$$

where Area_A is the area of the minimal surface in AdS_{d+2} whose boundary is ∂A; α_n is a n-dependent constant ($\lim_{n\to\infty} \frac{\alpha_n}{n}$ =finite).

This form (5.6) is almost clear from the following argument. First we notice that $\log\langle \Phi_n \rangle$ should be equal to the factor $\frac{1}{G_N^{(d+2)}}$ times a certain diffeomorphism invariant quantity as is clear from the supergravity side. Then the latter should have the momentum dimension $-d$. Only such a candidate is essentially the area term as in (5.6), assuming that it is given by a local integral.

Applying the formula (3.2) we find

$$S_A = \left(\left.\frac{\partial \alpha_n}{\partial n}\right|_{n=1}\right) \cdot \mathrm{Area}_A. \tag{5.7}$$

The coefficient can be fixed by requiring that the entanglement entropy at a finite temperature should be reduced to the thermal entropy (i.e. black hole entropy) when A is the total space (see also later discussions in section 7 on this point). This leads to (5.3).

6. Entanglement entropy in 2D CFT from AdS_3

We start with the AdS_3 ($d = 1$) in the global coordinates (5.2). According to AdS/CFT correspondence [6], the gravitational theories on this space are dual to $1 + 1$ dimensional

[13]Here we consider the AdS dual of the CFT with central charge nc. Finally we take the limit $n \to 1$.

[14]This can be understood from the general formula of two point functions [24, 25] (see also [56, 57]) and the behavior of the geodesic length (e.g. see (6.10)).

conformal field theories with the central charge [58]

$$c = \frac{3R}{2G_N^{(3)}}, \tag{6.1}$$

where $G_N^{(3)}$ is the Newton constant in three dimensional gravity[15].

6.1 AdS$_3$ space and UV cutoff in dual CFTs

At the boundary $\rho = \infty$ of the AdS$_3$, the metric is divergent. To regulate relevant physical quantities we need to put a cutoff ρ_0 and restrict the space to the bounded region $\rho \leq \rho_0$. This procedure corresponds to the ultra violet (UV) cutoff in the dual conformal field theory [25, 59]. If we define the dimensionless UV cutoff δ ($\propto$ *length*), then we find the relation $e^{\rho_0} \sim \delta^{-1}$. In the example of the previous section, δ should be identified with

$$e^{\rho_0} \sim \delta^{-1} = L/a. \tag{6.2}$$

Remember that L is the total length of the system and a is the lattice spacing (or UV cutoff). Notice that there is actually an ambiguity about the $\mathcal{O}(1)$ numerical coefficient in this relation[16].

The holographic principle tells us that true physical degrees of freedom of the gravitational theory in some region is represented by its boundary of that region. This is well-known in the black hole geometries and it leads to the celebrated area law of the Bekenstein-Hawking entropy. In the context of AdS/CFT correspondence degrees of freedom in AdS$_{d+1}$ space are represented by its boundary of the form $\mathbb{R}_t \times$ S^{d-1}, where the dual conformal field theory lives. We can compute the number of degrees of freedom N_{dof} by applying the area law in three dimensional spacetimes to the boundary in the AdS$_3$ space [59] . This leads to the following estimation

$$N_{\text{dof}} \sim \frac{\text{Boundary Length}}{4G_N^{(3)}} = \frac{2\pi R \sinh \rho_0}{4G_N^{(3)}} \simeq \frac{\pi c}{6} \cdot \frac{L}{a}. \tag{6.3}$$

The central charge c is roughly proportional to the number of fields. The ratio L/a counts the number of independent points in the presence of the lattice spacing a. Therefore the result (6.3) agrees with what we expect from the conformal field theory at least up to the unknown numerical coefficient.

6.2 Geodesics in AdS$_3$ and entanglement entropy in CFT$_2$

In the global coordinate of AdS$_3$ (5.2), the $1+1$ dimensional spacetime, in which the CFT$_2$ is defined, is identified with the cylinder $(t, \theta (\equiv \Omega_1))$ at the (regularized) boundary $\rho = \rho_0$. Then we consider the AdS dual of the setup in section 3.3. The subsystem A corresponds to $0 \leq \theta \leq 2\pi l/L$ and we can discuss the entanglement entropy by applying

[15] Remember that $G_N^{(d+2)}$ is defined such as $S_{\text{gravity}} = \frac{1}{16\pi G_N^{(d+2)}} \int d^{d+2}x\sqrt{g}R + \cdots$. for any dimension d.

[16] However, this ambiguity does not affect universal quantities which do not depend on the cut off a and we will consider such quantities in the later arguments.

our proposal (5.3). In this lowest dimensional example, the minimal surface γ_A, which plays the role of the holographic screen[17], becomes one dimensional. In other words, it is the geodesic line which connects the two boundary points at $\theta = 0$ and $\theta = 2\pi l/L$ with t fixed (see figure 4).

Then to find the entropy we calculate the length of the geodesic line γ_A. The geodesics in AdS$_{d+2}$ spaces are given by the intersections of two dimensional hyperplanes and the AdS$_{d+2}$ in the ambient $\mathbb{R}^{2,d+1}$ space such that the normal vector at the points in the intersections is included in the planes. The explicit form of the geodesic in AdS$_3$, expressed in the ambient $\vec{X} \in \mathbb{R}^{2,2}$ space, is

$$\vec{X} = \frac{R}{\sqrt{\alpha^2-1}} \sinh(\lambda/R) \cdot \vec{x} + R\left[\cosh(\lambda/R) - \frac{\alpha}{\sqrt{\alpha^2-1}} \sinh(\lambda/R)\right] \cdot \vec{y}, \qquad (6.4)$$

where $\alpha = 1 + 2\sinh^2\rho_0 \sin^2(\pi l/L)$; x and y are defined by

$$\begin{aligned} \vec{x} &= (\cosh\rho_0 \cos t, \cosh\rho_0 \sin t, \sinh\rho_0, 0), \\ \vec{y} &= (\cosh\rho_0 \cos t, \cosh\rho_0 \sin t, \sinh\rho_0 \cos(2\pi l/L), \sinh\rho_0 \sin(2\pi l/L))\,. \end{aligned} \qquad (6.5)$$

The length of the geodesic can be found as

$$\text{Length} = \int ds = \int d\lambda = \lambda_*, \qquad (6.6)$$

where λ_* is defined by

$$\cosh(\lambda_*/R) = 1 + 2\sinh^2\rho_0\ \sin^2\frac{\pi l}{L}. \qquad (6.7)$$

Assuming that the UV cutoff energy is large $e^{\rho_0} \gg 1$, we can obtain the entropy (5.3) as follows (using (6.1))

$$S_A \simeq \frac{R}{4G_N^{(3)}} \log\left(e^{2\rho_0}\sin^2\frac{\pi l}{L}\right) = \frac{c}{3}\log\left(e^{\rho_0}\sin\frac{\pi l}{L}\right). \qquad (6.8)$$

Indeed, this entropy exactly coincides with the known 2D CFT result (3.21), including the (universal) coefficients after we remember the relation (6.2).

6.3 Calculations in Poincare coordinates

It is useful to repeat the similar analysis in the Poincare coordinates (5.1). We pickup the spacial region (again call A) $-l/2 \leq x \leq l/2$ and consider its entanglement entropy as in section 3.2. We can find the geodesic line γ_A between $x = -l/2$ and $x = l/2$ for a fixed time t_0 (see also later analysis in section 7)

$$(x, z) = \frac{l}{2}(\cos s, \sin s), \qquad (\epsilon \leq s \leq \pi - \epsilon). \qquad (6.9)$$

[17]Notice that this is a codimension two space-like surface in spacetime following the definition in [12]. It may be more standard to call a holographic screen the codimension one hypersurface, which is a family of the codimension two surfaces, as in [10]. The latter is also called a screen hypersurface in [12].,holography,BiSu,Bousso

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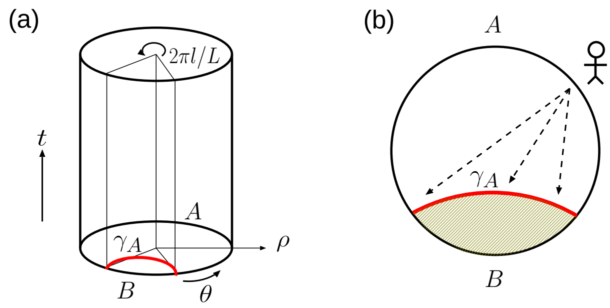

Figure 4: (a) AdS_3 space and CFT_2 living on its boundary and (b) a geodesics γ_A as a holographic screen.

The infinitesimal ϵ is the UV cutoff and leads to the cutoff z_{UV} as $z_{\mathrm{UV}} = \frac{l\epsilon}{2}$. Since $e^{\rho} \sim x^i/z$ near the boundary, we find $z \sim a$. The length of γ_A can be found as

$$\mathrm{Length}(\gamma_A) = 2R \int_{\epsilon}^{\pi/2} \frac{ds}{\sin s} = -2R \log(\epsilon/2) = 2R \log \frac{l}{a}. \tag{6.10}$$

Finally the entropy can be obtained as follows

$$S_A = \frac{\mathrm{Length}(\gamma_A)}{4G_N^{(3)}} = \frac{c}{3} \log \frac{l}{a}. \tag{6.11}$$

This again agrees with the well-known result (3.13) as expected.

6.4 Entropy on multiple disjoint intervals

Next we proceed to more complicated examples. Assume that the system A consists of multiple disjoint intervals. The entanglement entropy can be computed as in (3.20). In the dual AdS_3 description, the region A corresponds to $\theta \in \cup_{i=1}^{N} [\frac{2\pi r_i}{L}, \frac{2\pi s_i}{L}]$ at the boundary. In this case it is not straightforward to speculate the holographic screen (or minimal surface) γ_A . However, the result in the $1+1$ dimensional conformal field theory (3.20) can be rewritten into the following simple form

$$S_A = \frac{1}{4G_N^{(3)}} \left[\sum_{i,j} \mathrm{Length}(r_j, s_i) - \sum_{i<j} \mathrm{Length}(r_j, r_i) - \sum_{i<j} \mathrm{Length}(s_j, s_i) \right], \tag{6.12}$$

where $\mathrm{Length}(A, B)$ denotes the length of the geodesic line between two boundary points A and B. This shows how we choose γ_A. It is a linear combination of geodesic lines. Their coefficients are either 1 or -1. Thus some of the coefficients turn out to be negative [18]. One may also think that the surface which is just the union of the N geodesic line

[18] One may think the presence of minus signs is confusing from the viewpoint of holographic screen. Instead we would like to regard this as a singular (or just complicated) behavior which is typical only in the lowest dimension. In higher dimensional cases, we do not seem to have such a problem when ∂A is compact. Notice also that the total sum (6.12) is always positive. If we replace the surface γ_A with D-branes or fundamental strings (remember the similarity to Wilson loops) , the minus sign is analogous to ghost branes introduced recently in [60].

between $\theta = \frac{2\pi r_i}{L}$ and $\theta = \frac{2\pi s_i}{L}$ is equally good for the choice of γ_A. However, if each contribution to the entropy is denoted by S_{A_i}, we can show $\sum_{i=1}^{N} S_{A_i} \geq S(A)$, which coincides with the subadditivity relation. Since the smallest area surface clearly gives the dominant contribution in the gravity description, the original one S_A is preferred. In this way, we can understand that such negative coefficients are necessary by considering the limit where s_i coincides with r_{i+1} and requiring it reproduces the result for $N-1$ intervals.

6.5 Finite temperature cases

Next we consider how to explain the entanglement entropy (3.25) at finite temperature $T = \beta^{-1}$ from the viewpoint of AdS/CFT correspondence. Since we assumed that the spacial length of the total system L is infinite, we have $\beta/L \ll 1$. In such a high temperature circumstance, the gravity dual of the conformal field theory is described by the Euclidean BTZ black hole [61]. Its metric looks like

$$ds^2 = (r^2 - r_+^2)d\tau^2 + \frac{R^2}{r^2 - r_+^2}dr^2 + r^2 d\varphi^2. \tag{6.13}$$

The Euclidean time is compactified as $\tau \sim \tau + \frac{2\pi R}{r_+}$ to obtain a smooth geometry. We also impose the periodicity $\varphi \sim \varphi + 2\pi$. By taking the boundary limit $r \to \infty$, we find the relation between the boundary CFT and the geometry (6.13)

$$\frac{\beta}{L} = \frac{R}{r_+} \ll 1. \tag{6.14}$$

The subsystem for which we consider the entanglement entropy is given by $0 \leq \varphi \leq 2\pi l/L$ at the boundary. Then by extending our proposal (5.3) to asymptotically AdS spaces, the entropy can be computed from the length of the space-like geodesic starting from $\varphi = 0$ and ending to $\varphi = 2\pi l/L$ at the boundary $r = \infty$ for a fixed time. To find the geodesic line, it is useful to remember that the Euclidean BTZ black hole at temperature T is equivalent to thermal AdS_3 at temperature $1/T$. This equivalence can be interpreted as a modular transformation in the boundary CFT [62]. If we define the new coordinates

$$r = r_+ \cosh\rho, \quad \tau = \frac{R}{r_+}\theta, \quad \varphi = \frac{R}{r_+}t, \tag{6.15}$$

then the metric (6.13) indeed becomes the one in the Euclidean Poincare coordinates with t replaced by it. Now the computation of the geodesic line is parallel with what we did in section 6.2. We only need to replace $\sinh\rho$ and $\sin t$ with $\cosh\rho$ and $\sinh t$. In the end we find (6.6) with λ_* is now given by

$$\cosh\left(\frac{\lambda_*}{R}\right) = 1 + 2\cosh^2\rho_0 \sinh^2\left(\frac{\pi l}{\beta}\right), \tag{6.16}$$

where we took into account the UV cutoff $e^{\rho_0} \sim \beta/a$. Then our area law (5.3) precisely reproduces the known CFT result (3.25). We can extend these arguments to the multi interval cases as in the zero temperature case. We again obtain the formula (6.12) from the CFT result (3.24).

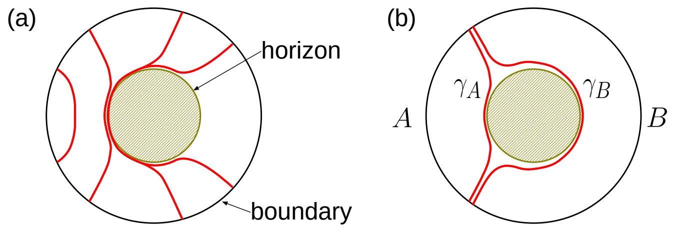

Figure 5: (a) Minimal surfaces γ_A in the BTZ black hole for various sizes of A. (b) γ_A and γ_B wrap the different parts of the horizon.

It is also useful to understand these calculations geometrically. The geodesic line in the BTZ black hole takes the form shown in figure 5(a). When the size of A is small, it is almost the same as the one in the ordinary AdS$_3$. As the size becomes large, the turning point approaches the horizon and eventually, the geodesic line covers a part of the horizon. This is the reason why we find a thermal behavior of the entropy when $l/\beta \gg 1$ in (3.26). The thermal entropy in a conformal field theory is dual to the black hole entropy in its gravity description via the AdS/CFT correspondence. In the presence of a horizon, it is clear that S_A is not equal to S_B (remember B is the complement of A) since the corresponding geodesic lines wrap different parts of the horizon (see figure 5(b)). This is a typical property of entanglement entropy at finite temperature as we mentioned in section 2.

6.6 Massive deformation

Now we would like to turn to $1+1$ dimensional massive quantum field theories. Such a theory can be typically obtained by perturbing a conformal field theory by a relevant perturbation. In the dual gravity side, this corresponds to a deformation of AdS$_3$ space. Since in the high energy limit the mass gap can be ignored, the deformation only takes place for small values $z < z_{\mathrm{IR}}$ of z in the Poincare coordinates. As in the well-known examples [63 – 65] in AdS$_5$, we expect the massive deformation caps off the end of the throat region.

Consider an $1+1$ dimensional infinite system divided into two semi-infinite pieces and define the subsystem A by one of them (i.e. $A = A_{\mathrm{SL}}$). Let us compute the entanglement entropy S_A in this setup. The important quantity in the massive theory is the correlation length ξ. This is identified with $\xi \sim z_{\mathrm{IR}}$ in the dual gravity side. Since we assumed that the subsystem A is infinite, we should take a geodesic (6.9) with a large value of $l (\gg \xi)$. The geodesic starts from the UV cutoff $z = a$ and ends at the IR cutoff $z = \xi$. Then we obtain the length of this geodesic as follows

$$\mathrm{Length}(\gamma_A) = \int_{\epsilon=2a/l}^{2\xi/l} \frac{ds}{\sin s} = R \log \frac{\xi}{a}. \tag{6.17}$$

In the end we find its entropy

$$S_A = \frac{\mathrm{Length}(\gamma_A)}{4G_N^{(3)}} = \frac{c}{6} \log \frac{\xi}{a}. \tag{6.18}$$

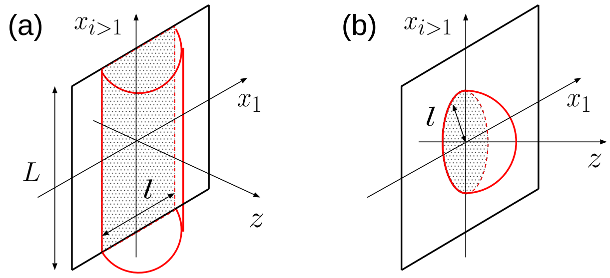

Figure 6: Minimal surfaces in AdS_{d+2}: (a) A_S and (b) A_D.

This perfectly reproduces the known result (3.27) in the $1+1$ dimensional quantum field theory.

7. Entanglement entropy in CFT_{d+1} from AdS_{d+2}

Since we have confirmed the proposed relation (5.3) in the lowest dimensional case $d = 1$, the next step is to examine higher dimensional cases. Our proposal (5.3) argues that the entanglement entropy in $d+1$ dimensional conformal field theories can be computed from the area of the minimal surfaces in AdS_{d+2} spaces. In the most of arguments in this section we employ the Poincare coordinates (5.1) for simplicity. Even though we cannot fully check our proposal due to the lack of general analytical results in the CFT side, we will manage to obtain some supporting evidences employing the previous results in section 4.

7.1 General results

For specific choices of the subsystem (or submanifold) A, it is easy to evaluate the area of minimal surfaces directly in AdS_{d+2} spaces of general dimensions d. Essentially this is possible by applying the techniques employed to compute the Wilson loops from AdS/CFT duality [66, 67, 56].

7.1.1 Entanglement entropy for straight belt A_S

First consider the entanglement entropy for the straight belt A_S (4.2) with the width l. The d dimensional minimal surface in AdS_{d+2} is given by minimizing the area functional (we set $x = x_1$ in the coordinate system (5.1))

$$\mathrm{Area} = R^d L^{d-1} \int_{-l/2}^{l/2} dx \frac{\sqrt{1 + (\frac{dz}{dx})^2}}{z^d}. \tag{7.1}$$

Regarding x as a time, we can find the Hamiltonian which does not depend on x. This leads to

$$\frac{dz}{dx} = \frac{\sqrt{z_*^{2d} - z^{2d}}}{z^d}, \tag{7.2}$$

where z_* is a constant. This equation determines the minimal surface γ_A (see figure 6(a)). Since $z = z_*$ is the turning point of the minimal surface, we require[19]

$$\frac{l}{2} = \int_0^{z_*} dz \frac{z^d}{\sqrt{z_*^{2d} - z^{2d}}} = \frac{\sqrt{\pi}\,\Gamma(\frac{d+1}{2d})}{\Gamma(\frac{1}{2d})} z_*. \tag{7.3}$$

Then the area is given by

$$\text{Area}_{A_S} = \frac{2R^d}{d-1}\left(\frac{L}{a}\right)^{d-1} - 2IR^d\left(\frac{L}{z_*}\right)^{d-1}, \tag{7.4}$$

where I is the constant

$$I = \frac{1}{d-1} - \int_0^1 \frac{dy}{y^d}\left(\frac{1}{\sqrt{1-y^{2d}}} - 1\right) = -\frac{\sqrt{\pi}\,\Gamma(\frac{1-d}{2d})}{2d\,\Gamma(\frac{1}{2d})} > 0. \tag{7.5}$$

In the end, we find the entanglement entropy from (5.3) using (7.3), (7.4) and (7.5)

$$S_{A_S} = \frac{1}{4G_N^{(d+2)}}\left[\frac{2R^d}{d-1}\left(\frac{L}{a}\right)^{d-1} - \frac{2^d\pi^{d/2}R^d}{d-1}\left(\frac{\Gamma(\frac{d+1}{2d})}{\Gamma(\frac{1}{2d})}\right)^d\left(\frac{L}{l}\right)^{d-1}\right], \tag{7.6}$$

Notice that the first divergent term is proportional to the area of ∂A i.e. L^{d-1} as we expect from the known area law in the field theory computations (2.8). The second term is finite and thus is universal (i.e. does not depend on the cutoff). This is the quantity which we can directly compare with the field theory counterpart. Notice that our result (7.6) does not include subleading divergent terms $\mathcal{O}(a^{-d+3})$. This is because A_S is in the straight shape. When we deform and bend it, the subleading divergent terms appear in general as we will see later in another example. For example, in the 4D case, the absence of log term is clear from the previous CFT analysis (4.34).

7.1.2 Entanglement entropy for circular disk A_D

Next we examine the case where subsystem A is given by the circular disk A_D (radius l) as defined in (4.3). We use the polar coordinate for $\mathbb{R}^d$ such that $\sum_{i=1}^d dx_i^2 = dr^2 + r^2 d\Omega_{d-1}^2$. The minimum surface is the d dimensional ball B^d defined by $z = z(r)$ (and Ω_{d-1} takes arbitrary values). The function $z(r)$ is found by minimizing the area functional

$$\text{Area}_{A_D} = R^d \cdot \text{Vol}(\mathrm{S}^{d-1}) \cdot \int_0^l dr r^{d-1} \frac{\sqrt{1 + (\frac{dz}{dr})^2}}{z^d}. \tag{7.7}$$

We can find the following simple solution from the equation of motion[20] for (7.7)

$$r^2 + z^2 = l^2. \tag{7.8}$$

[19] We employed the formula $\int_0^1 dx x^{\mu-1}(1-x^\lambda)^{\nu-1} = \frac{B(\mu/\lambda,\nu)}{\lambda}$, where $B(x,y) = \Gamma(x)\Gamma(y)/\Gamma(x+y)$.

[20] The equation of motion is given by $rzz'' + (d-1)z(z')^3 + (d-1)zz' + dr(z')^2 + dr = 0$.

Thus γ_A is a half of a d dimensional sphere (see figure 6(b)). This can be also found from the conformal map of the simplest case where ∂A is a single straight line (i.e. $A = A_{\rm SL}$) into A_D. Then we obtain its area

$$\begin{aligned}
\mathrm{Area}_{A_D} &= \mathrm{Vol}(\mathrm{S}^{d-1}) \cdot R^d \cdot \int_{a/l}^{1} dy \frac{(1-y^2)^{(d-2)/2}}{y^d} \\
&= \frac{2\pi^{d/2} R^d}{\Gamma(d/2)} \cdot \left[\frac{1}{d-1} \left(\frac{l}{a}\right)^{d-1} - \frac{d-2}{2(d-3)} \left(\frac{l}{a}\right)^{d-3} + \cdots \right].
\end{aligned} \tag{7.9}$$

In this expression (7.9), the omitted subleading terms $\cdots$ of the order $\mathcal{O}(a^{-d+5})$ include the logarithmic term $\sim \log \frac{l}{a}$ when d is odd. On the other hand, if d is even, the series end up with a constant term. Taking into account these, the final expression of the entanglement entropy can be found as follows applying (5.3)

$$\begin{aligned}
S_{A_D} &= \frac{2\pi^{d/2} R^d}{4G_N^{(d+2)} \Gamma(d/2)} \int_{a/l}^{1} dy \frac{(1-y^2)^{(d-2)/2}}{y^d} \\
&= p_1 \, (l/a)^{d-1} + p_3 \, (l/a)^{d-3} + \cdots \\
&\quad \cdots + \begin{cases} p_{d-1} \, (l/a) + p_d + \mathcal{O}(a/l), & d\text{: even,} \\ p_{d-2} \, (l/a)^2 + q \log{(l/a)} + \mathcal{O}(1), & d\text{: odd,} \end{cases}
\end{aligned} \tag{7.10}$$

where the coefficients are defined by

$$\begin{aligned}
p_1/C &= (d-1)^{-1}, \quad p_3/C = -(d-2)/[2(d-3)], \quad \cdots \\
p_d/C &= (2\sqrt{\pi})^{-1} \Gamma(d/2) \Gamma\left((1-d)/2\right) \quad (\text{if} \quad d = \text{even}), \\
q/C &= (-)^{(d-1)/2} (d-2)!!/(d-1)!! \quad (\text{if} \quad d = \text{odd}), \\
&\text{where} \quad C \equiv \frac{\pi^{d/2} R^d}{2G_N^{d+2} \Gamma(d/2)}.
\end{aligned} \tag{7.11}$$

We notice that the result (7.11) includes a leading UV divergent term $\sim a^{-d+1}$ and its coefficient is proportional to the area of the boundary ∂A as expected from the area law [16, 17] in the field theories (2.8). We have also subleading divergence terms which reflects the form of the boundary ∂A.

In particular, we prefer a physical quantity that is independent of the cutoff (i.e. universal). The final term in (7.11) has such a property. When d is even, it is given by a constant p_d. This seems to be somewhat analogous to the topological entanglement entropy (or quantum dimension) recently introduced in $2+1$ D topological field theories [22, 23], though our theory is not topological.

On the other hand, when d is odd, the coefficient of the logarithmic term $\sim \log(l/a)$ is universal as was so in the 2D case (3.13). Indeed, we found such a term in the analysis of 4D conformal field theories e.g. (4.29), which is proportional to the central charge. This issue will also be discussed in detail later.

This result is based on an explicit calculation when $A = A_D$. However, from the paper [56], we find that the behavior (7.10) is also true for any compact submanifold A with different coefficient p_k and q depending on the shape of A.

7.1.3 Multiple loops

When the system A consists of M disconnected regions (we call them $A_1, A_2, \cdots, A_M$), we need to find the minimal surface γ_A whose boundary ∂A is $A_1 \cup A_2 \cup \cdots \cup A_M$. If the distance between A_is are small enough we may find a connected surface of this property with a smaller area than those of the trivial disconnected ones. The connected one with the smallest area gives the dominant contribution in the supergravity partition function (see section 5.2) even when there are several other disconnected minimal surfaces.

However, if they are far apart, γ_A can be separated into several pieces as pointed out by [68] in the analogous problem of Wilson loop computations. Even if we take into account this complexity, the inequality (subadditivity) $S(A) \leq S(A_1) + S(A_2) + \cdots + S(A_N)$ is clearly satisfied.

It is also useful to consider a singular limit of such a multiple component case, i.e. when the subsystem A consists of the multiple straight belts $A_{S(1)}, A_{S(2)}, \cdots, A_{S(N)}$. In this situation, we can naturally obtain the entanglement entropy from the formula (6.12) by replacing the geodesic distance with the area of the minimal surfaces. This agrees with the free field computation which is a straightforward generalization of the result in section 4.1.1.

7.2 Entanglement entropy in $\mathcal{N} = 4$ SYM from AdS$_5\times$ S^5

So far we have discussed low energy gravity theories on AdS$_{d+2}$ and have not been careful about its high energy completion as quantum gravity. To understand the holographic relation better including the various quantum corrections, it is necessary to realize a concrete embedding into string theory. The most important such example is the AdS$_5\times$S^5 background in type IIB string theory. This background preserves the maximal 32 supersymmetries and is considered to be dual to the $\mathcal{N} = 4$ SU(N) Super Yang-Mills theory [6]. The supergravity approximation corresponds to the large t' Hooft coupling $\lambda = N g_{\rm YM}^2 \gg 1$ (i.e. strongly coupled) region. The planar limit $N \to \infty$ is equivalent to the weakly coupled region $g_s \to 0$ of type IIB string. Since we perform the supergravity analysis, the dual gauge theory is strongly coupled and the large N limit is taken.

The 5D Newton constant $G_N^{(5)}$ is given in terms of the 10D one

$$G_N^{(10)} = \frac{\kappa^2}{8\pi} = 8\pi^6 \alpha'^4 g_s^2, \tag{7.12}$$

as follows

$$G_N^{(5)} = \frac{G_N^{(10)}}{R^5 \mathrm{Vol}(\mathrm{S}^5)} = \frac{G_N^{(10)}}{\pi^3 R^5}. \tag{7.13}$$

The radius R of AdS$_5$ and S^5 is expressed as

$$R = (4\pi g_s \alpha'^2 N)^{\frac{1}{4}}. \tag{7.14}$$

Plugging these values (7.13) and (7.14) into the previous results (7.6) and (7.10), we obtain the following prediction of entanglement entropies in $\mathcal{N} = 4$ SU(N) super Yang-Mills theory

$$S_{A_S} = \frac{N^2L^2}{2\pi a^2} - 2\sqrt{\pi}\left(\frac{\Gamma\left(\frac{2}{3}\right)}{\Gamma\left(\frac{1}{6}\right)}\right)^3 \frac{N^2L^2}{l^2}, \tag{7.15}$$

$$S_{A_D} = N^2\left[\frac{l^2}{a^2} - \log\left(\frac{l}{a}\right) + \mathcal{O}(1)\right]. \tag{7.16}$$

Notice that these are proportional to N^2 as expected since the number of fields in the SU(N) gauge theory is proportional to N^2. Interestingly, (7.15) does not depend on $g^2_{\mathrm{YM}} = \frac{g_s}{2\pi}$.

Let us examine the first result (7.15). We notice that it has the same functional form as in the free field theories (4.11). Since the second term in (7.15) is finite, it is interesting to compare its coefficient with that of the free field theory result. The finite term in (7.15) is numerically expressed as

$$S^{\mathrm{Sugra}}_{A_S}|_{\mathrm{finite}} \simeq -0.0510 \cdot \frac{N^2L^2}{l^2}. \tag{7.17}$$

On the other hand, in the free field theory side we can employ the estimations (4.12). The $\mathcal{N}=4$ super Yang-Mills consists of a gauge field A_μ, six real scalar fields ($\phi^1, \phi^2, \cdots, \phi^6$) and four Majorana fermions ($\psi^1_\alpha, \psi^2_\alpha, \psi^3_\alpha, \psi^4_\alpha$). As we explained in section 4.1.3, the contribution from the gauge field is the same as those from two real scalar fields. In this way the total entropy in the free Yang-Mills theory is the same as those from 8 real scalars and 4 Majorana fermions . Thus we obtain from (4.12) the following estimation

$$S^{FreeYM}_{A_S}|_{\mathrm{finite}} \simeq -(8\times 0.0049 + 4\times 0.0097)\cdot\frac{N^2L^2}{l^2} = -0.078\cdot\frac{N^2L^2}{l^2}. \tag{7.18}$$

We observe that the free field result is larger than that in the gravity dual by a factor $\sim \frac{3}{2}$. This deviation is expected since the computation of the entanglement entropy[21] includes non-BPS quantities due to the anti-periodic boundary condition of fermions which appears when we compute the partition function on $n-$sheeted manifold M_n. This situation is very similar to the computation of thermal entropy [54], where we have a similar discrepancy (so-called $\frac{4}{3}$ problem). The fact that the discrepancy is of order one also in our computation can be thought as an encouraging evidence for our proposal. Also notice that the coefficient in the free Yang-Mills is larger than the one in the strongly coupled Yang-Mills. This is natural since the interaction of the form $\mathrm{Tr}[\phi_i,\phi_j]^2$ reduces the degrees of freedom [7].

Next we turn to our second result (7.16). In addition to the area law divergence, it includes a logarithmic term, whose coefficient is universal. This qualitative dependence of the entropy (7.16) on l agrees with our previous result from the Weyl anomaly (4.29). We will discuss the coefficient in front of the logarithmic term in more detail in the next subsection.

[21] As we notice in section 4.2, some parts of entanglement entropy are proportional to the central charges. They remain the same under exactly marginal deformation (e.g. changing coupling g_{YM}) since central charges do so.

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7.3 Entanglement entropy and central charges in 4D CFT from AdS$_5$

We can extend the previous computation to more general (i.e. less supersymmetric) conformal backgrounds by replacing S^5 with a compact five dimensional Einstein manifold X_5. The radius R of AdS$_5$ and X_5 is given by [69, 51]

$$R = \left(\frac{4\pi^4 g_s \alpha'^2 N}{\mathrm{Vol}(X_5)}\right)^{\frac{1}{4}}, \tag{7.19}$$

where N is again the number of D3-branes (or rank of the gauge group). The volume Vol(X_5) of X_5 is known to be inversely proportional to the central charge a [51]. Note that $a = c$ always holds when a CFT has its gravity dual of the form AdS$_5 \times X_5$. In the $\mathcal{N} = 4$ SU(N) super Yang-Mills theory the central charge is given by $a_{\mathcal{N}=4} = \frac{N^2-1}{4} \simeq \frac{N^2}{4}$.

The entanglement entropy S_A in general 4D CFTs of this type is given in terms of $S_A^{\mathcal{N}=4}$ in $\mathcal{N} = 4$ SU(N) super Yang-Mills theory

$$S_A = \left(\frac{a}{a_{\mathcal{N}=4}}\right) \cdot S_A^{\mathcal{N}=4}, \tag{7.20}$$

i.e. S_A is proportional to the central charge a. This is naturally understood by considering that the central charge a measures degrees of freedom in the 4D CFT. Notice that here we are assuming a strongly coupled 4D CFT in order to apply the AdS/CFT duality. In our previous CFT analysis done in section 4.2, we have only shown that a part of entanglement entropy is proportional to the central charge (4.29).

As we have seen, the coefficient of the logarithmic term (called γ_2 in section 4.2) in (7.16) is universal and is given by the central charge a times a numerical factor. Thus it is very interesting to compare the factor between the gauge theory and the gravity. When the 2D surface ∂A is generic (with a finite size) and the background is an arbitrary asymptotically AdS$_5$ space, the logarithmic term in the area of the minimal surface γ_A can be found from the general formula given in [56]. This leads to

$$l\frac{d\mathrm{Area}(\partial A)}{dl}|_{\mathrm{finite}} = \int_{\partial A} d^2x\sqrt{g}\left(-\frac{1}{8}|H|^2 - \frac{1}{4}g^{\alpha\beta}R_{\alpha\beta} + \frac{1}{12}R\right), \tag{7.21}$$

where α and β are the coordinates which are tangent to ∂A and orthogonal to i, j directions; H is the mean curvature. As we did in section 4.2, we work below under the special assumption that the second fundamental forms are zero to make arguments simple. Then we can show

$$\begin{aligned} R &\simeq R_{\Sigma=\partial A} + 2R_{ii} - R_{ijij}, \\ g^{\alpha\beta}R_{\alpha\beta} &\simeq R_{\Sigma=\partial A} + R_{ii} - R_{ijij}. \end{aligned} \tag{7.22}$$

We can also neglect $|H|^2$ term in (7.21). In the end, we can rewrite (7.21) into the following form

$$l\frac{d\mathrm{Area}(\partial A)}{dl}|_{\mathrm{finite}} = \int_{\partial A} d^2x\sqrt{g}\left(\frac{1}{6}R_{ijij} - \frac{1}{12}R_{ii} - \frac{1}{6}R_{\Sigma=\partial A}\right). \tag{7.23}$$

By considering the setup dual to the 4D $\mathcal{N}=4$ SU(N) super Yang-Mills theory ($a = c \simeq \frac{N^2}{4}$), it is straightforward to check that the gravity result (7.23) agrees[22] with the previous result (4.34) obtained from the Weyl anomaly[23]. It will be an interesting future problem to examine terms which include the second fundamental forms and check the complete agreement.

7.4 Entanglement entropy from $\mathbf{AdS}_{4,7}\times\mathbf{S}^{7,4}$ in M-theory

Other important supersymmetric examples of AdS spaces are $\mathrm{AdS}_4\times\mathrm{S}^7$ and $\mathrm{AdS}_7\times\mathrm{S}^4$ in eleven dimensional supergravity (or M-theory). They preserve the maximal 32 supersymmetries. They are considered to be dual to 3D $\mathcal{N}=8$ SCFT and 6D $(2,0)$ SCFT, respectively [6, 7]. They are obtained from the world-volume theories on M2 and M5-branes (or strongly coupled limit of D2 and D4-branes). The numbers of the branes are denoted by N. Since these theories have not been completely understood due to the strongly coupled problem, it will be very useful to compute any new physical quantities.

The 11D Newton constant $G_N^{(11)}$ is given in terms of 11D plank length l_p as follows[24]

$$(2\pi)^8 l_p^9 = 16\pi G_N^{(11)} = 2\kappa_{11}^2. \tag{7.24}$$

Let us first discuss the $\mathrm{AdS}_4\times\mathrm{S}^7$ example. The radius of AdS_4 and S^7 are

$$2R_{\mathrm{AdS}_4} = R_{S^7} = l_p(32\pi^2 N)^{\frac{1}{6}}. \tag{7.25}$$

The four dimensional Newton constant can be found after the compactification on S^7

$$G_N^{(4)} = \frac{48\pi^3 l_p^9}{R_{S^7}^7}. \tag{7.26}$$

Then we find the following entanglement entropy defined for the straight belt A_S

$$S_{A_S} = \frac{\mathrm{Area}}{4G_N^{(4)}} = \frac{\sqrt{2}}{3}N^{3/2}\left[\frac{L}{a} - \frac{4\pi^3}{\Gamma(1/4)^4}\frac{L}{l}\right]. \tag{7.27}$$

The entropy for the circular disk A_D we find

$$S_{A_D} = \frac{\mathrm{Area}}{4G_N^{(4)}} = \frac{\sqrt{2}\pi}{3}N^{3/2}\left[\frac{l}{a} - 1\right]. \tag{7.28}$$

Notice that the constant terms in (7.27) and (7.28) are universal. The dependence $\sim N^{3/2}$ of degrees of freedom is typical in the 3D $\mathcal{N}=8$ SCFT.

[22]Under this assumption we cannot deal with the circular disk case A_D because the second fundamental forms are non-zero (i.e. $\Gamma^i_{\alpha\beta}\neq 0$). However, it is possible to see that the contribution (4.31), which comes from the topological term $\tilde{R}^2$, coincides with the gravity result. Indeed we expect that the other contribution from the Weyl tensor term W^2 is vanishing since the disk is conformally equivalent to the straight line, in which case there is no log term (notice also that the W^2 term is a conformal invariant).

[23]The derivation of the Weyl anomaly from the AdS/CFT duality was first done in [69].

[24]Our convention is such that $S_{11Dsugra} = \frac{1}{2\kappa_{11}^2}\int d^{11}x[\sqrt{g}R+\cdots]$. We follow the convention in [7].

In the AdS$_7\times$ S^4 case, in addition to (7.24), we have

$$R_{\mathrm{AdS}_7} = 2R_{S^4} = 2l_p(\pi N)^{\frac{1}{3}}, \tag{7.29}$$

and

$$G_N^{(7)} = \frac{6\pi^5 l_p^9}{R_{S^4}^4}. \tag{7.30}$$

Then we find the following results

$$S_{A_S} = \frac{2}{3\pi^2}N^3\left[\frac{L^4}{a^4} - 16\pi^{5/2}\left(\frac{\Gamma(3/5)}{\Gamma(1/10)}\right)^5\frac{L^4}{l^4}\right]. \tag{7.31}$$

$$S_{A_D} = \frac{32}{9}N^3\left[\frac{1}{4}\cdot\frac{l^4}{a^4} - \frac{3}{4}\cdot\frac{l^2}{a^2} + \frac{3}{8}\log(l/a)\right]. \tag{7.32}$$

Notice that the constant term in (7.31) and the coefficient of $\log(l/a)$ in (7.32) are universal. The overall dependence $\sim N^3$ is again peculiar to 6D $(2,0)$ SCFT.

7.5 Finite temperature case

Consider the $\mathcal{N}=4$ super Yang-Mills theory on $\mathbb{R}^4$ at finite temperature T. This system is dual to the AdS black hole geometry [25, 70]

$$ds^2 = R^2\left[\frac{du^2}{hu^2} + u^2\left(-hdt^2 + dx_1^2 + dx_2^2 + dx_3^2\right) + d\Omega_5^2\right], \tag{7.33}$$

where

$$h = 1 - \frac{u_0^4}{u^4}, \quad u_0 = \pi T. \tag{7.34}$$

Various analyzes show that this theory has properties of a confining gauge theory [70, 7].

We would like to compute the entanglement entropy for the straight line in this model. The subsystem A is defined by $t=$ fixed, $-l/2 \leq x(\equiv x_1) \leq l/2$, $u\to\infty$, and $x_2, x_3=$arbitrary. The regularized volume in the x_2 and x_3 direction is denoted by L^2. Then the area is given by[25]

$$\mathrm{Area} = R^3L^2\int_{-l/2}^{l/2} dx u^3\sqrt{1+\frac{u'^2}{u^4-u_0^4}}. \tag{7.35}$$

We can integrate the equation of motion as

$$\frac{du}{dx} = \sqrt{(u^4-u_0^4)(u^6/u_*^6-1)}. \tag{7.36}$$

We require

$$\frac{l}{2} = \int_{u_*}^{\infty} du\frac{1}{\sqrt{(u^4-u_0^4)(u^6/u_*^6-1)}}, \tag{7.37}$$

[25] This system is very similar to the one that appears in the computation of Wilson loop at finite temperature [71].

where $u_*(> u_0)$ is the value of $u(x)$ at the turning point $x = 0$. Using (7.36) we can rewrite (7.35)

$$\text{Area} = 2R^3L^2 \int_{u_*}^{\infty} du \frac{u^6}{\sqrt{(u^4 - u_0^4)(u^6 - u_*^6)}}. \tag{7.38}$$

As usual, (7.38) contains the UV divergent term which is proportional to a^{-2}. However, we are interested in the term which is peculiar to this kind of confining gauge theory. Indeed we can find that in the large l limit (i.e. $u_* \sim u_0$), the main contribution (except the UV divergence) of the integrals (7.37) and (7.38) comes from the region near $u = u_*$, which leads to the relation

$$\text{Area}_{\text{finite}} \sim \pi^3 N^2 R^3 L^2 l T^3. \tag{7.39}$$

Thus we obtain the finite part (i.e. we subtracted the UV divergent terms) of the entropy in this limit

$$S_{\text{finite}} = \frac{\pi^2 N^2}{2} T^3 L^2 l = \frac{\pi^2 N^2}{2} T^3 \times \text{Area}(A). \tag{7.40}$$

The important point is that this entropy (7.40) is proportional to the area of not ∂A but A as opposed to the area law term (5.3). Thus it is extensive as in the thermal entropy. This agrees with the field theory side since the entanglement entropy should include the thermal entropy contribution as is obvious from its definition. In the gravity side, it occurs because γ_A wraps a part of the black hole horizon and thus (7.40) is equal to the fraction of black hole entropy, which shows the thermal behavior. This means that the behavior of the entanglement entropy is rather different before and after the cofiniment/de-confinement transition when we consider the $\mathcal{N} = 4$ super Yang-Mills on $\mathbb{R} \times \mathrm{S}^3$ (see [72] and references therein for recent studies of this phase transition). Thus the entanglement entropy plays an role similar to an "order parameter". Other geometrical properties are also parallel with the AdS_3 case as figure 5 shows.

7.6 Massive deformations

As a final example we would like to discuss the entanglement entropy in $d+1$ dimensional massive QFTs. Typically we can obtain such theories by considering massive deformations of a $d+1$ dimensional CFT. In principle, this can be done by looking at supergravity solutions dual to non-conformal field theories such as [64, 63, 65]. Instead here we approximate the geometry simply by cutting off the IR region $z > \xi$ of the AdS_{d+2} space as we did in the $d = 1$ case. Here ξ is the correlation length and we are assuming $\xi \gg l$. [26]

7.6.1 Straight belt A_S

Let us start with the computation of the entanglement entropy for the straight belt in a massive theory by the simple method explained in the above. This leads to the following

[26] When a quantum ground state of a massive theory has non-trivial Berry phases, contribution from the Berry phase to the entanglement entropy is also important [73].

estimation

$$
\begin{aligned}
S_{A_S} &= 2\frac{R^d L^{d-1}}{4G_N^{(d+2)}} \times \int_a^\xi dz \frac{\sqrt{\left(\frac{dx}{dz}\right)^2+1}}{z^d} = \frac{L^{d-1}R^d}{2G_N^{(d+2)} z_*^{d-1}} \int_{a/z_*}^{\xi/z_*} \frac{d\lambda}{\lambda^d\sqrt{1-\lambda^{2d}}} \\
&= \frac{R^d L^{d-1}}{2G_N^{(d+2)}} \Bigg[-\frac{a^{-d+1}}{-d+1} + \frac{\xi^{-d+1}}{-d+1} \\
&\quad + \frac{1}{2}\frac{1}{d+1}\frac{\xi^{d+1}}{z_*^{2d}} + \cdots + \frac{(2n-1)!!}{(2n)!!}\frac{1}{(2n-1)d+1}\frac{\xi^{(2n-1)d+1}}{z_*^{2nd}} + \cdots \Bigg] \\
&= \frac{R^d L^{d-1}}{2G_N^{(d+2)}} \left[\frac{a^{-d+1}}{d-1} - \frac{\xi^{-d+1}}{d-1} + r_1 \frac{\xi^{d+1}}{l^{2d}} + \cdots + r_n \frac{\xi^{(2n-1)d+1}}{l^{2nd}} + \cdots \right],
\end{aligned} \tag{7.41}
$$

where r_is are some numerical constants. We assumed the same form of the minimal surface as in the conformal case and thus the relation between z_* and l is the same as before (7.3).

7.6.2 Circular disk A_D

Next we examine the entanglement entropy for the circular disk A_D (radius l) in a massive theory. We assume the same minimal surface $r^2 + z^2 = l^2$ as in the conformal case.

$$
\begin{aligned}
S_{A_D} &= \frac{R^d \mathrm{Vol}(S^{d-1})}{4G_N^{(d+2)}} \int dr\, r^{d-1} \frac{\sqrt{1+(\frac{dz}{dr})^2}}{z^d} \\
&= \frac{2\pi^{d/2} R^d}{4\Gamma(d/2) G_N^{(d+2)}} \int_{a/l}^{\xi/l} dy \frac{\left(1-y^2\right)^{(d-2)/2}}{y^d}.
\end{aligned} \tag{7.42}
$$

The integral in the final expression in (7.42) has the following series expansion when d is even (we set $d = 2n$)

$$
\begin{aligned}
&\int_{a/l}^{\xi/l} dy\, \frac{(1-y^2)^{(d-2)/2}}{y^d} \\
&= \Bigg[-\frac{1}{d-1}\left(\frac{l}{\xi}\right)^{d-1} + \frac{d-2}{2(d-3)}\left(\frac{l}{\xi}\right)^{d-3} + \cdots \\
&\quad \cdots - \frac{(-1)^n}{2^n n!} \frac{(d-2n)(d-2n+2)\cdots(d-2)}{d-2n-1}\left(\frac{l}{\xi}\right)^{d-2n-1} + \cdots \Bigg] \\
&\quad + \Bigg[\frac{1}{d-1}\left(\frac{l}{a}\right)^{d-1} - \frac{1}{2}\frac{d-2}{d-3}\left(\frac{l}{a}\right)^{d-3} + \cdots \\
&\quad \cdots + \frac{(-1)^n}{2^n n!} \frac{(d-2n)(d-2n+2)\cdots(d-2)}{d-2n-1}\left(\frac{l}{a}\right)^{d-2n-1} \Bigg],
\end{aligned} \tag{7.43}
$$

where the expansion of a/l is truncated since we take the limit $a \to 0$ in the final expression.

When d is odd ($d = 2n+1$), we obtain the same result (7.43) except that we have to be careful about the two terms $\mathcal{O}\left((l/\xi)^{d-2n-1}\right)$ and $\mathcal{O}\left((l/a)^{d-2n-1}\right)$ in (7.43) which are proportional to $\frac{1}{d-2n-1} \to \infty$. The divergences are canceled out and produce a log term

$$(-1)^{\frac{d-1}{2}} \frac{(d-2)!!}{(d-1)!!} \log \frac{\xi}{a}. \tag{7.44}$$

Thus in the odd d case, we just have to replace the two terms in (7.43) with (7.44). Note that this term has the same coefficient as the one in the conformal case, i.e. q/C in (7.11) in our approximation. In summary we find

$$\begin{aligned} S_{A_D} &= \frac{2\pi^{d/2}R^d}{4\Gamma(d/2)G_N^{(d+2)}} \left[\frac{1}{d-1} \frac{l^{d-1}}{a^{d-1}} + (\text{subleading divergences } \mathcal{O}(l^{d-3}/a^{d-3})) \right] \\ &+ \frac{2\pi^{d/2}R^d}{4\Gamma(d/2)G_N^{(d+2)}} \left[-\frac{1}{d-1} \frac{l^{d-1}}{\xi^{d-1}} + \mathcal{O}(l^{d-3}/\xi^{d-3}) \right] \\ &+ \begin{cases} 0 & d\text{: even,} \\ (-1)^{\frac{d-1}{2}} \dfrac{2\pi^{d/2}(d-2)!!R^d}{4\Gamma(d/2)(d-1)!!G_N^{(d+2)}} \cdot \log \dfrac{\xi}{a} & d\text{: odd.} \end{cases} \end{aligned} \tag{7.45}$$

7.7 Entanglement entropy in some non-conformal theories

The best way to derive the entanglement entropy in massive (or non-conformal) theories is to start with their dual supergravity backgrounds instead of the previous crude approximation. Since usually such backgrounds include complicated metric and many other fields, we would like to make a first step by looking at some simple cases such as the near horizon limit of Dp-branes. Here we would like to examine the example of the D2-branes and NS5-branes. It will be an interesting future problem to analyze more complicated but more realistic examples.

7.7.1 D2-branes case

The decoupling limit of supergravity solution for N D2-branes is given by the following metric and dilaton [74]

$$\begin{aligned} ds^2 &= \alpha' \left(\frac{U^{5/2}}{g_{\mathrm{YM}}\sqrt{6\pi^2 N}}(-dx_0^2 + dx_1^2 + dx_2^2) + \frac{g_{\mathrm{YM}}\sqrt{6\pi N^2}}{U^{5/2}} dU^2 + \frac{g_{\mathrm{YM}}\sqrt{6\pi^2 N}}{U^{1/2}} (d\Omega_6)^2 \right), \\ e^{2\phi} &= g_{\mathrm{YM}}^2 \left(\frac{g_{\mathrm{YM}}\sqrt{6\pi^2 N}}{U^{5/2}} \right)^{1/2}. \end{aligned} \tag{7.46}$$

This supergravity background (7.46) is dual to the field theory limit of world-volume theory on N D2-branes. This field theory is described by the 3D SU(N) super Yang-Mills theory with the dimensionful coupling constant $g_{\mathrm{YM}}(\propto \text{energy}^{1/2})$. The radial direction U is proportional to the energy scale in this field theory.

To avoid the strongly coupled region $g_s \gg 1$ and the high curvature region $\alpha' R \gg 1$, we trust the supergravity solution (7.46) when [74]

$$g_{\mathrm{YM}}^2 N^{1/5} \ll U \ll g_{\mathrm{YM}}^2 N. \tag{7.47}$$

The other two regions $U \gg g_{\rm YM}^2 N^{1/5}$ and $U \ll g_{\rm YM}^2 N$ are well described by the 3D superconformal field theory (or M2-branes) and the weakly coupled Yang-Mills theory, respectively.

Under this condition (7.47), we would like to compute the entanglement entropy holographically in the straight belt case $A = A_S$. First we notice that the dilaton is not constant and thus the definition of $G_N^{(4)}$ is not clear. However, it is easy to find a natural extension of our formula (5.3) by remembering the relation $\frac{1}{G_N^{(4)}} = \frac{1}{G_N^{(10)}} \int_{S^6} d^6x\sqrt{g}$. Consider the following functional for any 2D surface γ_A such that $\partial\gamma_A = A$

$$\frac{1}{4G_N^{(10)}} \int_{\gamma_A \times S^6} d^8x \; e^{-2\phi}\sqrt{g}, \tag{7.48}$$

and try to minimize it. This procedure singles out what should be called a minimal surface γ_A. It is trivial to see that this procedure is reduced to the original relation (5.3) when the dilaton is constant.

Then we find that γ_A is defined by (the notation is the same as in section 7.1.1)

$$\frac{dU}{dx} = \frac{U^{5/2}}{g_{\rm YM}\sqrt{6\pi^2 N}}\sqrt{\frac{U^7}{U_*^7} - 1}, \tag{7.49}$$

where U_* is the turning point of the surface and we assume $g_{\rm YM}^2 N^{1/5} \ll U_*$. Following the same way of analysis in section 7.1, in the end we obtain the entanglement entropy

$$S_{A_S} = \frac{NLU_0^2}{5\pi g_{\rm YM}^2} - c \cdot \frac{N^{5/3}L}{(g_{\rm YM})^{2/3} l^{4/3}}, \tag{7.50}$$

where U_0 is the UV cutoff (assuming $U_0 \ll g_{\rm YM}^2 N$), and $c = \frac{1}{5}\left(\frac{4\sqrt{2}}{\sqrt{3}}\right)^{4/3} \pi^{3/2} \left(\frac{\Gamma(5/7)}{\Gamma(3/14)}\right)^{7/3}$. The first term is proportional to the length of γ_A (i.e. L) in (7.50) and is an analogue of the area law divergence term[27]. The second term is interesting since it is finite and depends on l, non-trivially. Its N dependence $\propto N^{5/3}$ is between the free field result N^2, and the IR fixed point result $N^{3/2}$ (see (7.27)) of the 3D $\mathcal{N} = 8$ superconformal field theory, as expected. As in the 4D case, we learn that the Yang-Mills interaction reduces the degree of freedom.

7.7.2 NS5-branes case

The throat part of N NS5-branes is described by the following well-known metric [75, 74]

$$\begin{aligned} ds^2 &= -dx_0^2 + \sum_{i=1}^{5} dx_i^2 + N\alpha' \frac{dU^2}{U^2} + N\alpha'(d\Omega_3)^2, \\ e^{\phi} &= \left(\frac{(2\pi)^3 N}{g_{\rm YM}^2 U^2}\right)^{1/2}. \end{aligned} \tag{7.51}$$

[27] It is proportional to the square of the cut off energy and is different from the area law relation (2.8). However, this is not any contradiction because we cannot set $U_0 \to \infty$ due to the constraint (7.47). In such a high energy region, we cannot neglect the stringy corrections and it is better to use the weakly coupled Yang-Mills description.

We assume type IIB string theory to fix notations. To take the decoupling limit, we keep the Yang-Mills coupling $g_{\mathrm{YM}}^2 = (2\pi)^3\alpha'$ finite and take the limit $g_s \to 0$. This leads to the little string theory (for a review see [76]). Notice that this theory is not a local field theory and shows non-local behaviors such as the Hagedorn transition.

The calculation of the entanglement entropy can be done as before. However, in this case[28] of NS5-branes, we encounter the following unusual feature. Consider the straight line case and try to find solutions for the minimal surface equation

$$\frac{dU}{dx} = \sqrt{\frac{(2\pi)^3}{Ng_{\mathrm{YM}}^2}U^2\left(\frac{U^4}{U_*^4} - 1\right)}. \tag{7.52}$$

Smooth solutions are allowed only for a fixed value of l_*

$$l_* = \int_{U_*}^{\infty}\left(\frac{dx}{dU}\right)dU = \frac{\sqrt{Ng_{\mathrm{YM}}^2}}{4\sqrt{2\pi}} = \frac{\pi}{2}\sqrt{N\alpha'}. \tag{7.53}$$

This suggests a phase transition at the energy scale $(l_*)^{-1}$. Indeed, the value $l^{-1} \sim \frac{1}{\sqrt{N\alpha'}}$ is the order of the Hagedorn temperature T_H in the little string theory[29]. At least, we can claim from the computation in (7.53) that there is no solution when $l < l_*$. We can understand this because the lack of locality means that we cannot define the entanglement entropy when the size of A becomes the same order of T_H^{-1}.

8. Conclusions and discussions

In this paper we presented detailed discussions of the holographic interpretation of the entanglement entropy proposed in our earlier letter [1]. We gave a derivation of our proposal (5.3) in the $\mathrm{AdS}_3/\mathrm{CFT}_2$ case by applying the basic computation [24, 25] of correlation functions in AdS/CFT correspondence. As for the higher dimensional case, we are still lacking its complete derivation from standard AdS/CFT correspondence, even though we offered an intuitive explanation and several non-trivial evidences of our proposal (5.3). This deserves further investigations. The proof of the strong subadditivity (2.6) will also be a non-trivial test for this purpose, say.

The application of the proposal (5.3) to various quantum field theories is also intriguing. Since we mainly analyzed the conformal field theories, it would be useful to compute the entanglement entropy in massive theories. In this paper, we did a rough approximation by cutting off the IR region by hand and also analyzed simple non-conformal backgrounds of D2-branes and NS5-branes. The next step will be to compute the entanglement entropy by considering supergravity backgrounds dual to more realistic massive theories such as 4D confining theories. There we expect that the entanglement entropy can be used as an alternative of the Wilson loop to distinguish the confinement. Indeed, we already noticed that the behavior of entanglement entropy is drastically changed before and after the

[28] Via the S-duality the analysis of the D5-branes leads to the same result.

[29] The holographic entanglement entropy in this case takes the form of $S_{A_S} = c_1 \cdot \frac{NL^4}{g_{YM}{}^2}U_0^2 - c_2 L^4 N^2 \frac{U_*^2}{l^2}$, where c_1 and c_2 are a certain constant.

deconfinement phase transition in $\mathcal{N} = 4$ super Yang-Mills theory at finite temperature from the AdS_5 side. Also, we obtained a singular behavior of the entanglement entropy in the background with many NS5-branes, which will probably be related to the non-locality or the Hagedorn transition in the little string theory.

We also investigated the properties of the entanglement entropy from the conformal field theory side. Especially we showed that important parts of entanglement entropy are proportional to central charges in any 4D CFTs from the analysis of Weyl anomaly. Even though we did not find this property for the other parts of the entropy, which are invariant under the Weyl scaling, the holographic analysis tells us that the total entanglement entropy in strongly coupled 4D CFTs is proportional to the central charge a. These facts offer us an evidence that the central charge is proportional to degrees of freedom in a given conformal field theory. It would be an interesting future problem to study the relation between possible c-theorems in more than two dimensions and the property of entanglement entropy.

Several aspects of the entanglement entropy revealed in this paper can have many implications on (strongly interacting) QFTs, some of which might be realized in condensed matter physics, say. For example, we derived the scaling of entanglement entropy (7.10) for a compact submanifold A based on AdS/CFT correspondence where the coefficients p_d and q are universal and conformal invariant. We expect that this is a generic feature which might be applicable for systems that does not necessarily have gravity (AdS) description. Thus, it is interesting to investigate these quantities in several strongly interacting systems at criticality. In a sense, these quantities are a generalization of the central charges in CFTs in even spacetime dimensions, or the quantum dimension in topological field theories. (Note also that there is no counter part of the central charges in odd spacetime dimensions.) For example, at least in principle, we can numerically study these universal quantities in the entanglement entropy in gapless spin liquids, and compare them with those computed from several candidate effective field theories [18]. Also, even though these effective field theories are suspected to be a gauge theory, it might not be straightforward to identify the Wilson loop operator in a generic microscopic spin model. In that situation, one can instead look at the entanglement entropy since our analysis for AdS black holes suggests it can be at least as useful as the Wilson loop.

Finally, our computation of entanglement entropy may also be useful to uncover holographical duals of string theory backgrounds which are not well-understood, such as de-Sitter spaces and Gödel spaces[30]. This is because the entanglement entropy captures the basic degrees of freedom in the dual theory and because it can be easily estimated classically in the gravity side.

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[30]Recent discussions on these spaces from this viewpoint can be found e.g. in [77] and [78].

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